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European Options and Local Times

Master's thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Technology in the Degree Programme in Engineering Physics and Mathematics.

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ABSTRACT OF
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Modern mathematical finance is based on the methods of stochastic analysis and usually models apply martingale theory and stochastic integration theory. Specially, the well-known Black and Scholes model is based on Itô integrals and geometric Brownian motion.

This study establishes the connection between Black and Scholes model and local time of geometric Brownian motion. First we introduce the Black and Scholes model and derive the basic properties of the market model. Another objective is to find a new integral representation for local time of geometric Brownian motion through the Black and Scholes model. We also study some applications.

The results of the study are promising. Besides the new integral representation for local time, we derive two ways to compute the expectation of the local time of geometric Brownian motion. We applicate the integral representation to Black and Scholes differential equation and study the local time of exponential martingale. We also derive a formula for the price of European options which are determined by the difference of two convex functions. As an application of this, we show that in order to hedge a convex European option, the capital needed increases as the maturity of an option increases.

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Moderni rahoitusteoria perustuu stokastisen analyysin menetelmiin ja usein mallit soveltavat martingaaliteoriaa ja stokastista integraaliteoriaa. Erityisesti Black-Scholes malli perustuu Itô-integraaleihin ja geometriseen Brownin liikkeeseen.

Tässä työssä tutkitaan Black-Scholes mallin ja geometrisen Brownin liikkeen lokaalin ajan välistä yhteyttä. Ensin työssä esitellään Black-Scholes malli ja todistetaan markkinamalliin liittyvät perustulokset. Seuraavaksi työssä etsitään uusi integraaliesitys geometrisen Brownin liikkeen lokaalille ajalle Black-Scholes mallin avulla. Työssä käsitellään myös sovelluksia.

Työn tulokset ovat lupaavia. Uuden integraaliesityksen lisäksi työssä esitellään kaksi eri tapaa laskea geometrisen Brownin liikkeen lokaalin ajan odotusarvo. Integraaliesitystä sovelletaan Black-Scholes differentiaaliyhtälöön ja eksponentiaalisen martingaalin lokaaliin aikaan. Työssä myös johdetaan hinta Eurooppalaisille optioille, jotka määräytyvät kahden konveksin funktion erotuksena. Tämän sovelluksena työssä näytetään että mitä suurempi konveksin Eurooppalaisen option maturiteetti on, sitä enemmän pääomaa vaaditaan siltä suojautumiseen.

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After five years of studying, the journey is about to end and a boy grown to a man receives the honourable title of Master of Science.

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Lauri Viitasaari

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Chapter 1

Introduction

1.1 History

The history of stochastic integration began with Brownian motion named after R. Brown (1827). The mathematical studies of Brownian motion can be traced to three independent sources, T.N. Thiele, L. Bachelier and A. Einstein, and often the credit for the mathematical model of Brownian motion is given to Einstein for his work in 1905. Einstein assumed that Brownian motion is a process with continuous paths and independent stationary Gaussian increments, which is similar to the modern definition of Brownian motion. There have been many mathematicians such as I. Gihman and N. Wiener who have had an influence on the birth of stochastic integration and analysis, but still Kiyoshi Itô is regarded as the father of stochastic integration. The first paper on stochastic integration from Itô was published in 1944. Itô's construction of stochastic integral was later extended by J.L. Doob in 1953 and the stochastic calculus has been developed further since. (Jarrow & Protter, 2004)

Mathematical finance is a field of science which combines stochastic analysis and economics. One of the starting points of mathematical finance is the PhD of H.M. Markowitz in 1952, even though Bachelier studied the stock prices of the Paris stock market in his time at the beginning of the 20th century. One of the first who started to apply stochastic calculus in finance was R.C. Merton in 1969. With the help of Merton, F. Black and M. Scholes developed the well-known pricing formula for European options.

1.2 Economical background

Prices of products are determined by the laws of economics. If the price of the product grows higher, then there are fewer people interested in buying such a product and thus the quantity on market decreases. This is called the law of demand. On the other hand, the law of supply states that as the demand of the product increases, so does the price. This is true, because when the market price of a product increases, then there are more people interested in selling a product and thus the quantity supplied becomes higher. The law of demand and supply determines an equilibrium at the market such that the price of a product is a certain equilibrium price and the quantity demanded equals the quantity supplied. When the price of the product is higher than the equilibrium price, then there is more supply than demanded. The competition leads to lower prices and market price falls into the equilibrium. Vice versa, when the price is lower than the equilibrium price, there is more demand than is supplied. Then the people are willing to pay more to get the product resulting price to increase into the equilibrium. However, recent events and studies have shown that these laws may sometimes fail. A good example of this is the current financial crisis. The fear that prices of stocks fall down gets people to sell their ownings and this accelerates the fall of prices. In this case, the increase in supply causes prices to fall, not to increase as it should according to the law of supply. In contract, the increase of demand cause prices to increase also. Despite of this fact, laws of supply and demand works in case of regular products such as an oil barrel or a bottle of milk.

An arbitrage opportunity is a way of doing profit without risk. For example, consider two different banks where the other offers higher interest rate for savings than the other for loans. Then one could take a loan from the second bank and save the money into the first one. As a result, one gains without any risk. Economically arbitrage cannot exists, since the market forces should eliminate such opportunities. For an illustrative example, consider a case where one can buy a product in a market and sell it at higher price in another market. Then no one would buy the product from the last one, since it is cheaper on the first market. On the other hand, no one would sell the product at cheaper price, since the seller gets more profit at the second market. As a result, the laws of economics force prices to an equilibrium and the arbitrage vanishes.

Nowadays banks do business with very complicated claims such as options, forwards, etc. An option is an agreement which gives the owner the right, but not the obligation to execute the option. For example, an option can be a right to sell or buy a stock at certain price. We consider specially European type of options, which can be executed only at the expiration date determined by the agreement. The purpose of the trading is to do profit, but since arbitrage cannot exists, the trading involves risks which can be enormous. For example, consider a case where an option gives the owner right to buy a stock at certain price at expiration date and the stock price increases enormously. The seller of the option loses money, for she is obliged to sell the stock at lower price than the current market price. This kind of option is called the European call option.

The trading at the market involves risks and when there is significant capital involved, organizations have to do some risk management. One way to reduce the risk is hedging. The purpose of the hedging is to construct a replicating portfolio, which contains other claims that turn out to be profitable, if the original claim is unprofitable and some of the capital invested is hedged. Of course, the replicating portfolio is unprofitable if the original claim is profitable and thus the hedging reduces the possible profit also. In a complete hedge of a claim the value of the replicating portfolio is precisely the value of the original claim.

1.3 Research objectives

The objective of this study is to examine the connection between Black and Scholes model and local time of geometric Brownian motion. The first objective is to review the details of Black and Scholes model. The second objective is to find a new integral representation for the local time of geometric Brownian motion through the Black and Scholes model. We also consider some applications.

In chapter 2, we construct stochastic integral and introduce basic mathematical concepts. Moreover, we introduce the local time of a semimartingale and Tanaka's formula. In chapter 3, we introduce the Black and Scholes model and prove the basic properties of the model. Specially, we concentrate on European type claims. Chapter 4 contains actual results. We derive the integral representation for the local time of geometric Brownian motion. We apply the representation to a few cases and analyse the Black and Scholes differential equation and the local time of exponential martingale. We also derive a formula for the price of European type options which are determined by a difference of two convex functions.

Chapter 2

Mathematical preliminaries

In this chapter we introduce mathematical preliminaries for further use. First we recall the basic consepts and results of stochastics in section 2.1. In section 2.2, we define the stochastic integral and recall the main results such as Itô's formula and the martingale property of the integral. We also explain, how and when the integral can be interprepted in a pathwise manner, which gives an economical meaning to integrals. In section 2.3, we introduce the consept of local time of a semimartingale and Tanaka's formula.

2.1 Basic concepts

We assume that the reader is familiar with basics of stochastics and measure theory. We refer to Jacod & Protter, 2004 and Rudin, 1987 for more information. Specially, the reader should be familiar with Lebesgue dominated convergence theorem and Fubini's theorem for σ-finite measures (Rudin, 1987).

First we recall some basic definitions.

Definition 2.1.1. *A continuous time process X in $(\Omega,\mathcal{F},\mathbb{P})$ is a collection $(X)_{t\geq 0}$ of random variables. We say that, with fixed ω, the mapping $t \longrightarrow X_t(\omega)$ is the path of the process. The process is continuous if it has continuous paths almost surely.*

Definition 2.1.2. *Let $\mathcal{F}$ be a σ-algebra and X a random variable. X is independent of $\mathcal{F}$ if for any Borel-measurable B and $F \in \mathcal{F}$*

$$\mathbb{P}(\{X \in B\} \cap F) = \mathbb{P}(X. \in B)\mathbb{P}(F).$$

Definition 2.1.3. *A probability space with history is a space* (Ω, $\mathcal{F}$,$\mathbb{F}$,$\mathbb{P}$)*, where we associate a standard probability space* (Ω, $\mathcal{F}$,$\mathbb{P}$) *with a family* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ *of sub-*σ*-algebras of* $\mathcal{F}$*, where* $\mathbb{F}$ *is increasing i.e.* $\mathcal{F}_s \subset \mathcal{F}_t$ *whenever* $s \leq t$*. The family* $\mathbb{F}$ *is called a history.*

Remark. The history can be interprepted as the information available up to t. The history generated by a continuous time process X is a family $\mathbb{F} = (\mathcal{F})_{t\geq 0}$ of sub-σ-algebras such that $\mathcal{F}_t = \sigma(X_s, s \leq t)$. In this case, the history is the information generated by the process X. This can be interprepted as that we know the values of random variables X_s for all $s \leq t$ i.e. we know the path of the process up to time t.

Definition 2.1.4. *A process* X *is adapted to history* $\mathbb{F}$ *if* X_t *is* $\mathcal{F}_t$*-measurable* $\forall t$*. We use short notation* $X_t \in \mathcal{F}_t$ *if* X_t *is* $\mathcal{F}_t$*-measurable and* $X \in \mathbb{F}$ *if* X *is adapted to* $\mathbb{F}$*.*

Remark. If X is adapted to $\mathbb{F}$, then the information $\mathcal{F}_t$ contains the information how the path of the process behaves up to t.

Definition 2.1.5. *A predictable* σ*-algebra* $\mathcal{P}(\mathbb{F})$ *is the smallest* σ*-algebra, which makes all left-continuous* $\mathbb{F}$*-adapted processes* X *measurable. A process* X *is predictable, if it is* $\mathcal{P}(\mathbb{F})$*-measurable.*

Definition 2.1.6 (Brownian motion)**.** *A continuous time stochastic process* $(W)_{t\geq 0}$ *is a Brownian motion if*

- $W_t - W_s$ *is independent of the* σ*-algebra* $\mathcal{F}_s^W$ $\forall s \leq t$*,*
- $W_t - W_s \sim N(0, |t-s|)$*,*
- *The paths of the process* W *are continuous.*

Remark. The first property means that the Brownian motion has independent increments. There are several equivalent definitions in mathematical literature.

The definition of the Brownian motion is axiomatic and does not require the existence of such a process. It turns out that such a process can be constructed and hence does exists (see Revuz & Yor, 1999).

Definition 2.1.7 (Wiener space)**.** *Probability space with history* (Ω, $\mathcal{F}$,$\mathbb{F}$,$\mathbb{P}$) *is a Wiener space if*

- $\Omega = C([0,T])$,
- $\mathcal{F}_t = \sigma(s \to \omega(s), s \leq t)$,
- $\mathcal{F} = \mathcal{F}_T$,
- *$\mathbb{P}$ is a probability measure such that the process $W_t : (\omega, t) \to \omega(t)$ is a Brownian motion.*

Remark. The first property means that the paths of the processes are continuous functions on $[0,T]$. Second and fourth property together means that the history is Brownian history i.e. generated by Brownian motion. In other words, Wiener space is simply a probability space with history, where paths of processes are continuous functions on $[0,T]$ and the measure is supported by the paths of Brownian motion.

We need to construct a stochastic integral to model the stock prices. In order to do so, we recall some basic concepts of the martingale theory. We refer to Jacod & Protter, 2004 and Revuz & Yor, 1999 for details.

Definition 2.1.8. *Let (Ω, $\mathcal{F}$,$\mathbb{P}$) be a probability space, $X \in L^1(\mathbb{P})$ and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. The conditional expectation of X with respect to $\mathcal{G}$ is a unique random variable $Y \in \mathcal{G}$, denoted by $\mathbb{E}[X|\mathcal{G}]$, such that*

$$\int_G X \mathrm{d}\mathbb{P} = \int_G Y \mathrm{d}\mathbb{P} \quad \forall G \in \mathcal{G}.$$

Remark. Define an indicator function of a measurable set A as usual:

$$I_A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases}.$$

Then the conditional expectation Y is the unique random variable satisfying

$$\mathbb{E}[X I_G] = \mathbb{E}[Y I_G] \quad \forall G \in \mathcal{G}.$$

Lemma 2.1.9. *Let X,Z be integrable random variables and $Y = \mathbb{E}[X|\mathcal{G}]$. Then the following properties are valid.*

- *Conditional expectation is linear.*

- $\mathbb{E}[Y] = \mathbb{E}[X]$.
- *If* $X \in \mathcal{G}$, *then* $Y = X$.
- *If* $\mathcal{H} \subset \mathcal{G}$, *then* $\mathbb{E}[Y|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$.
- *If* X *is independent of* $\mathcal{G}$, *then* $Y = \mathbb{E}[X]$.
- *If* $Z \in \mathcal{G}$ *and* $XZ \in L^1$, *then* $\mathbb{E}[ZX|\mathcal{G}] = Z\mathbb{E}[X|\mathcal{G}]$.
- *(Jensen) If* f *is convex and* $f(X) \in L^1$, *then* $f(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[f(X)|\mathcal{G}]$.
- *Monotone convergence theorem, Lebesgue dominated convergence theorem and Fatou's lemma are valid for conditional expectation.*

Definition 2.1.10. *An integrable, adapted process* M *is a martingale on* $[0, T]$ *with respect to history* $\mathbb{F}$, *if*

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \text{ a.s. } \forall 0 \leq s \leq t \leq T. \tag{2.1}$$

Lemma 2.1.11. *Assume that* M *is a martingale on* $[0, T]$. *Then*

$$\mathbb{E}[M_t] = \mathbb{E}[M_0]$$

for every $t \in [0, T]$.

Proof. By Lemma 2.1.9 and the martingale property we obtain

$$\mathbb{E}[M_t] = \mathbb{E}[\mathbb{E}(M_t|\mathcal{F}_0)] = \mathbb{E}[M_0].$$

□

Definition 2.1.12. *An integrable, adapted process* M *is a submartingale with respect to history* $\mathbb{F}$, *if*

$$\mathbb{E}[M_t|\mathcal{F}_s] \geq M_s \text{ a.s. } \forall s \leq t$$

and supermartingale with respect to history $\mathbb{F}$, *if*

$$\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s \text{ a.s. } \forall s \leq t.$$

Definition 2.1.13. *A partition $\pi_n[s,t]$ of the interval $[s,t]$ is the set $\{t_1, t_2, \ldots, t_n\}$ where $s = t_1 < t_2 < \cdots < t_n = t$.*

Definition 2.1.14. *Define*

$$\mathcal{V}_{[s,t]}(Y) := \sup \sum_{t_i \in \pi_n[s,t]} |Y_{t_{i+1}} - Y_{t_i}|,$$

where the supremum is taken over all partitions $\pi_n[s,t]$. The process Y is called a process with bounded variation on the interval $[s,t]$, if

$$\mathcal{V}_{[s,t]}(Y) < \infty.$$

Definition 2.1.15. *A random variable $\theta : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ is a stopping time with respect to history $\mathbb{F}$, if $\{\theta \leq t\} \in \mathcal{F}_t \quad \forall t$.*

Definition 2.1.16. *An integrable, adapted process M is a local martingale, if there exists an increasing sequence θ_n of stopping times such that $\theta_n \to \infty$ as $n \to \infty$ and a stopped process defined by*

$$M_t^{\theta_n} = M_{\min(t,\theta_n)} I_{\theta_n > 0}$$

is a martingale.

One of the most powerful results in martingale theory is Doob-Meyer decomposition theorem. The theorem states that under certain assumptions, each submartingale X has an unique decomposition

$$X_t = X_0 + M_t + A_t,$$

where M is a martingale, A is an adapted, increasing process and $M_0 = A_0 = 0$. This gives a motivation to the next definition.

Definition 2.1.17. *A continuous adapted process X is a continuous semimartingale if it has a decomposition*

$$X_t = X_0 + M_t + A_t, \tag{2.2}$$

where M is a continuous local martingale and A is a continuous process with locally bounded variation. Moreover, we have $M_0 = A_0 = 0$.

Remark. Due to Jordan decomposition (Rudin, 1987) a function has bounded variation if and only if it is a difference of two non-decreasing functions. Thus every increasing and decreasing function has bounded variation.

Remark. By Doob-Meyer decomposition, every sub- and supermartingale can be represented as the sum of a martingale and a process which is increasing with submartingales and decreasing with supermartingales. Thus, every sub- and supermartingale is a semimartingale.

Definition 2.1.18. *The quadratic variation of the continuous process Y is a process*

$$< Y >_t = < Y, Y >_t := \lim_{n \to \infty} \sum_{t_i \in \pi_n[0,t]} (Y_{t_{i+1}} - Y_{t_i})^2 \tag{2.3}$$

if the limit exists in probability.

Remark. The quadratic variation process exists for all continuous semimartingales.

Definition 2.1.19. *A process M is a square integrable martingale on $[0, T]$ if it is a martingale on $[0, T]$ and*

$$\mathbb{E}[M_t^2] < \infty \quad \forall t \in [0, T].$$

Lemma 2.1.20. *Let M be a continuous square integrable martingale on $[0, T]$. Then $M^2 - < M, M >$ is a continuous martingale on $[0, T]$.*

Proof. We refer to Revuz & Yor, 1999 for details. □

Remark. Since $M^2 - < M, M >$ is a martingale and M is square integrable, we have that $< M, M >$ is integrable.

Example. Brownian motion is adapted to its own history and integrable by definition. From the definition of Brownian motion and by properties of conditional expectation it follows that

$$\begin{aligned} \mathbb{E}[W_t | \mathcal{F}_s] &= \mathbb{E}[W_t - W_s + W_s | \mathcal{F}_s] \\ &= \mathbb{E}[W_t - W_s | \mathcal{F}_s] + W_s \\ &= \mathbb{E}[W_t - W_s] + W_s = W_s. \end{aligned}$$

Hence, Brownian motion is a martingale with respect to its own history.

Example. Assume that $\mathcal{F}$ is a σ-algebra, X is $\mathcal{F}$-measurable and Y is independent of $\mathcal{F}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and

$$g(x) = \mathbb{E}[f(x, Y)].$$

Let Z be a bounded $\mathcal{F}$-measurable random variable. By definition

$$g(x) = \int_{\mathbb{R}} f(x, y)\mathbb{P}(Y \in \mathrm{d}y)$$

and since Y is independent of $\mathcal{F}$,

$$\mathbb{P}(X \in \mathrm{d}x, Z \in \mathrm{d}z, Y \in \mathrm{d}y) = \mathbb{P}(X \in \mathrm{d}x, Z \in \mathrm{d}z)\mathbb{P}(Y \in \mathrm{d}y).$$

Now if f is bounded enough such that we may apply Fubini's theorem (see Rudin, 1987), we can change the order of integrations and obtain

$$\begin{aligned}
\mathbb{E}[f(X,Y)Z] &= \int_{\mathbb{R}^3} f(x,y) z \mathbb{P}(X \in \mathrm{d}x, Z \in \mathrm{d}z, Y \in \mathrm{d}y) \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f(x,y)\mathbb{P}(Y \in \mathrm{d}y)\right) z\mathbb{P}(X \in \mathrm{d}x, Z \in \mathrm{d}z) \\
&= \int_{\mathbb{R}^2} g(x) z \mathbb{P}(X \in \mathrm{d}x, Z \in \mathrm{d}z) \\
&= \mathbb{E}[g(X)Z].
\end{aligned}$$

Next we let $F \in \mathcal{F}$ be arbitrary and choose $Z = I_F$. From this we conclude that, by the definition of conditional expectation,

$$\mathbb{E}[f(X,Y)|\mathcal{F}] = g(X). \tag{2.4}$$

We use this result later in section 3.4.

2.2 Stochastic integrals

The Black and Scholes model assumes that the stock price follows geometric Brownian motion and thus we give the definition first.

Definition 2.2.1 (Geometric Brownian motion). *Geometric Brownian motion is a* *process S_t satisfying stochastic differential equation*

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t, S_0 = S_0, \tag{2.5}$$

where $\sigma > 0$ and $\mu \in \mathbb{R}$ are constants.

We know that the paths of the Brownian motion are not differentiable (Revuz & Yor, 1999) and therefore the last term does not make sense. However, we can define a stochastic integral and interpret stochastic differential equations as integral equations. For example, we interpret equation (2.5) as an integral equation

$$S_T = S_0 + \int_0^T \mu S_t \mathrm{d}t + \int_0^T \sigma S_t \mathrm{d}W_t. \tag{2.6}$$

For B-S model, we need to define stochastic integral with respect to a standard Brownian motion given by Definition 2.1.6 and with respect to geometric Brownian motion on the interval $[s, t]$, where $0 \leq s < t \leq T < \infty$. Time T represents the maturity of an option and thus the assumption that T is finite is realistic. The usual way of defining stochastic integral is the Itô's construction. In Itô's construction we define a simple predictable process as

$$H = \sum_{k=1}^{n} \alpha_k I_{(\theta_{k-1}, \theta_k]},$$

where $\alpha_k \in \mathcal{F}_{\theta_{k-1}}$ and $\theta_k, k = 1, \ldots, n+1$ is an increasing sequence of non-negative finite stopping times. Next we define an integral of this simple H with respect to a continuous square integrable martingale M on $[0, T]$ by

$$(HoM)_t = \sum_{k=1}^{n} \alpha_k (M_{\min(\theta_k, t)} - M_{\min(\theta_{k-1}, t)}).$$

This integral is also a continuous square integrable martingale on $[0, T]$. Next we take more general predictable process H_t. If the condition

$$\mathbb{E}\left[\int_0^T H_s^2 \mathrm{d} < M, M >_s\right] < \infty$$

is satisfied, then there exists a sequence H^n of simple left-continuous processes such that

$$\mathbb{E}\left[\int_0^T (H_s - H_s^n)^2 \mathrm{d} < M, M >_s\right] \Rightarrow 0$$

and we define the integral of H with respect to M as L^2-limit of integrals of these simple predictable processes. We also have the following theorem.

Theorem 2.2.2. *Assume that H satisfies*

$$\mathbb{E}\left[\int_0^T H_s^2 \mathrm{d} < M, M >_s\right] < \infty. \tag{2.7}$$

Then the integral process

$$N_t = \int_0^t H_s \mathrm{d} M_s \tag{2.8}$$

is continuous square integrable martingale on $[0, T]$. We also have the Itô isometry

$$\mathbb{E}\left[\left(\int_0^t H_s \mathrm{d} M_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 \mathrm{d} < M, M >_s\right]. \tag{2.9}$$

Moreover,

$$\mathbb{E}[N_t] = 0 \quad \forall t \in [0, T].$$

Proof. Clearly $N_0 = 0$. Thus the last property follows from the martingale property and Lemma 2.1.11. Let N_t^n denote the approximating sum i.e.

$$N_t^n = \sum_k \alpha_k^n (M_{\min(\theta_k^n, t)} - M_{\min(\theta_{k-1}^n, t)}).$$

We know that the N_t^n is a square integrable martingale on $[0, T]$. Direct computation shows that the isometry holds also. Moreover, one can show that the condition (2.7) implies that N_t^n is a Cauchy sequence in the space of continuous square integrable martingales. We know also that the space of continuous square integrable martingales is complete. Thus the limit is also a continuous square integrable martingale. The isometry follows directly. □

Remark. If H satisfies the condition

$$\int_0^T H_s^2 \mathrm{d} < M, M >_s \quad < \infty \quad a.s., \tag{2.10}$$

but not the condition (2.7), then one can show that the integral process (2.8) is still a local martingale. This is done by defining an increasing sequence of stopping times by

$$\theta_k = \inf\{t : \int_0^T H_s^2 \mathrm{d} < M, M >_s \geq k\}.$$

Clearly, θ_k tends to infinity as $k \to \infty$. Now the stopped integral process $N_t^{\theta_k}$ defined as

$$N_t^{\theta_k} = \int_0^{\min(t,\theta_k)} H_s \mathrm{d}M_s$$

is a martingale. Moreover, one can check that

$$\int_0^{\min(t,\theta_k)} H_s \mathrm{d}M_s = \int_0^t H_s \mathrm{d}M_s^{\theta_k}.$$

Thus, the integral process (2.8) can be defined as the almost sure limit

$$\int_0^t H_s \mathrm{d}M_s = \lim_{k\to\infty} \int_0^t H_s \mathrm{d}M_s^{\theta_k}.$$

This is true also, when M is only a local martingale. Hence, stochastic integrals are usually local martingales.

The final step of Itô's construction is to define an integral of a process H with respect to a continuous semimartingale X with decomposition

$$X_t = X_0 + M_t + A_t.$$

This is simply defined by

$$\int_0^t H_s \mathrm{d}X_s = \int_0^t H_s \mathrm{d}M_s + \int_0^t H_s \mathrm{d}A_s.$$

Here the first integral is the stochastic integral with respect to a continuous martingale. The second is with respect to a continuous process with bounded variation and thus can be defined as Riemann-Stieltjes integral.

Now we have constructed the Itô integral with respect to a continuous semimartingale. The well-known Itô's formula states that the class of semimartingales is closed under transforms $F(t, X_t)$, if F is smooth enough.

Theorem 2.2.3 (Itô's formula). *Let X be a continuous semimartingale and $F \in C^{1,2}(\mathbb{R}_+, \mathbb{R})$. Then*

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t F_x(s, X_s)\mathrm{d}X_s \\ &+ \tfrac{1}{2}\int_0^t F_{xx}(s, X_s)\mathrm{d} < X, X >_s + \int_0^t F_t(s, X_s)\mathrm{d}s. \end{aligned} \tag{2.11}$$

Remark. In the case of Brownian motion, the quadratic variation is given by

$$< W, W >_t = t.$$

Thus the Itô's formula takes the form

$$\begin{aligned} F(t, W_t) &= F(0, W_0) + \int_0^t F_x(s, W_s)\mathrm{d}W_s \\ &+ \tfrac{1}{2}\int_0^t F_{xx}(s, W_s)\mathrm{d}s + \int_0^t F_t(s, W_s)\mathrm{d}s. \end{aligned}$$

We need also the stochastic integral with respect to geometric Brownian motion S_t. Motivated by the definition, we define the integral of H with respect to S by

$$\int_0^T H_t \mathrm{d}S_t = \int_0^T \mu H_t \mathrm{d}t + \int_0^T \sigma H_t \mathrm{d}W_t. \tag{2.12}$$

Consider next a stochastic differential equation

$$\frac{\mathrm{d}Z_t}{Z_t} = \sigma \mathrm{d}W_s$$

equivalently in integral form

$$Z_t = Z_0 + \int_0^t \sigma Z_s \mathrm{d}W_s.$$

By Itô's formula (2.20), we obtain that the unique solution is given by

$$Z_t = Z_0 e^{\sigma W_t - \frac{1}{2}\sigma^2 t}. \tag{2.13}$$

On the other hand, the stochastic integral is a martingale by Theorem 2.2.2. Hence, Z_t is a martingale. Z_t is called a stochastic exponent or an exponential martingale. By this we obtain that geometric Brownian motion defined as the solution to (2.5) is given by

$$S_t = S_0 e^{\mu t + \sigma W_t - \frac{1}{2}\sigma^2 t}. \tag{2.14}$$

The expected value of geometric Brownian motion is given by

$$\mathbb{E}[S_t] = S_0 e^{\mu t}.$$

For the quadratic variation of S it holds

$$\mathrm{d} < S, S >_t = \sigma^2 S_t^2 \mathrm{d}t.$$

This implies that if the condition

$$\mathbb{E}\left[\int_0^T H_u^2 S_u^2 \mathrm{d}u\right] < \infty \tag{2.15}$$

holds, then the integral process

$$\int_0^t H_u S_u \mathrm{d}W_u$$

is a martingale on $[0, T]$.

Example. Let us check that (2.15) holds when $H_t = 1$ identically. By Fubini's theorem, we only need to check that

$$\int_0^T \mathbb{E}\left[S_t^2\right] \mathrm{d}t < \infty.$$

Substituting (2.14) we obtain

$$\int_0^T \mathbb{E}\left[S_0^2 e^{2\mu t + 2\sigma W_t - \sigma^2 t}\right] \mathrm{d}t \leq S_0^2 \max_{0 \leq t \leq T} e^{2\mu t - \sigma^2 u} \int_0^T \mathbb{E}\left[e^{2\sigma W_t}\right] \mathrm{d}t.$$

Thus we only need to check that

$$\int_0^T \mathbb{E}\left[e^{2\sigma W_t}\right] \mathrm{d}t < \infty.$$

Recall that $T < \infty$. Thus, since $W_t \sim N(0,t)$, we have

$$\int_0^T \mathbb{E}\left[e^{2\sigma W_t}\right] \mathrm{d}t < \infty$$

by computations in appendix A. We also note that geometric Brownian motion is square integrable for every $t \in [0,T]$ with finite T. Thus, the quadratic variation process $< S, S >_t$ is integrable for every $t \in [0,T]$.

We will also need the fact that every continuous local martingale of bounded variation is constant almost surely. The proof of this is based on the following lemma.

Lemma 2.2.4. *Assume that A is a continuous process and has bounded variation on $[0,T]$. Then, for every t $\leq T$,*

$$\int_0^t A_s \mathrm{d}A_s = \frac{1}{2}(A_t^2 - A_0^2).$$

Proof. The fact that following Abel summation formula

$$A_t^2 - A_0^2 = 2\sum_{\pi_n[0,t]} A_{t_k}(A_{t_{k+1}} - A_{t_k}) + \sum_{\pi_n[0,t]} (A_{t_{k+1}} - A_{t_k})^2.$$

hold is easy to prove. Since A is continuous and has bounded variation,

$$\sum_{\pi_n[0,t]} (A_{t_{k+1}} - A_{t_k})^2 \leq \max |A_{t_{k+1}} - A_{t_k}| \mathcal{V}_{[0,t]}(A) \to 0$$

as $n \to \infty$. Thus we have

$$A_t^2 - A_0^2 = 2 \lim_{n\to\infty} \sum_{\pi_n[0,t]} A_{t_k}(A_{t_{k+1}} - A_{t_k})$$

and since this limit is the integral

$$\int_0^t A_s \mathrm{d}A_s,$$

we have the result. □

Corollary 2.2.5. *Assume that M is a continuous local martingale. Then it has bounded variation if and only if it is constant almost surely.*

Proof. If M is constant, then it has bounded variation. Assume next that M has bounded variation. Without loss of generality, we may assume that $M_0 = 0$. Then by Lemma 2.2.4

$$M_t^2 = 2\int_0^t M_s \mathrm{d}M_s.$$

Assume that $\sup_{s\in[0,t]} |M_s| \le K$ and $\mathcal{V}_{[0,t]}(M) \le K$. Then

$$|\sum_{\pi_n[0,t]} M_{t_k}(M_{t_{k+1}} - M_{t_k})| \le \sup_{s\in[0,t]} |M_t| \sum_{\pi_n[0,t]} |M_{t_{k+1}} - M_{t_k}| \le K^2.$$

Making use of Lemma 2.1.9 we obtain

$$\begin{aligned}
\textstyle\sum_{\pi_n[0,t]} \mathbb{E}\left[M_{t_k}(M_{t_{k+1}} - M_{t_k})\right] &= \textstyle\sum_{\pi_n[0,t]} \mathbb{E}\left(\mathbb{E}\left[M_{t_k}(M_{t_{k+1}} - M_{t_k})|\mathcal{F}_{t_k}\right]\right) \\
&= \textstyle\sum_{\pi_n[0,t]} \mathbb{E}\left(M_{t_k}\mathbb{E}\left[M_{t_{k+1}} - M_{t_k}|\mathcal{F}_{t_k}\right]\right) \\
&= 0.
\end{aligned}$$

Now the approximating sum is bounded and hence we can apply Lebesgue dominated convergence theorem. Thus we have

$$\mathbb{E}[M_t^2] = 2\mathbb{E}\left[\int_0^t M_s \mathrm{d}M_s\right] = 0$$

for every $t \le T$. Hence $M_t = 0$ almost surely. Next we define a stopping time θ_K as the first time when either $|M_t| > K$ or $\mathcal{V}_{[0,t]}(M) > K$. Then the claim holds for the stopped process

$$M_t^{\theta_K} = M_{\min(t,\theta_K)} I_{\theta_K > 0}$$

for every $K > 0$. Next we let $K \to \infty$. Then $\theta_K \to \infty$ and thus

$$M_t = \lim_{K\to\infty} M_t^{\theta_K}$$

and we have the result. □

We want that our model corresponds to reality as well as possible and therefore the model needs to be arbitrage-free. We also want that claims can be replicated in some sense. The B-S model fulfills both criterias. For this we introduce a version of Girsanov's theorem and Itô-Clark representation theorem.

Definition 2.2.6. *A measure Q is absolutely continuous with respect to $\mathbb{P}$, if for each set A with $\mathbb{P}(A) = 0$, $Q(A) = 0$ holds also. The measures are equivalent if Q is absolutely continuous with respect to $\mathbb{P}$ and $\mathbb{P}$ is absolutely continuous with respect to Q.*

Lemma 2.2.7 (Radon-Nikodym). *Assume that a measure Q is absolutely continuous with respect to $\mathbb{P}$. Then there exists a positive integrable random variable H such that*

$$Q(A) = \mathbb{E}_{\mathbb{P}}[HI_A]$$

for every measurable set A. Moreover, the process H is unique almost surely. The process H is called the Radon-Nikodym derivative.

Let $\mathbb{P}_t$ denote the restriction of the measure $\mathbb{P}$ into the σ-algebra $\mathcal{F}_t$. With this notation we have the following theorem of Girsanov.

Theorem 2.2.8 (Girsanov). *Assume that W is ($\mathbb{P}$,$\mathbb{F}$)-Brownian motion and $\mu \in \mathbb{R}$. Let*

$$Z_t^\mu = e^{-\mu W_t - \frac{1}{2}\mu^2 t}$$

and define a measure Q^μ absolutely continuous with respect to $\mathbb{P}$ such that Z_t^μ is the corresponding Radon-Nikodym derivative of Q_t^μ with respect to $\mathbb{P}_t$. Then Q^μ is equivalent to $\mathbb{P}$ and the process

$$W_t^\mu = W_t + \mu t \tag{2.16}$$

is (Q^μ,$\mathcal{F}$)-Brownian motion.

Theorem 2.2.9 (Itô-Clark). *Let W be Brownian motion and Y a continuous square integrable $\mathcal{F}_T^W$-measurable random variable. Then there exists a unique square integrable, predictable process H s.t.*

$$Y = \mathbb{E}[Y] + \int_0^T H_s \mathrm{d}W_s. \tag{2.17}$$

2.2.1 Pathwise interpretation of stochastic integrals

In mathematical finance, the Itô integral is difficult to interpret in economic meaning. One way to overcome this regret is to interpret stochastic integrals in a pathwise manner i.e. we want to define integrals as an almost sure limit of the Riemann-Stieltjes sum. Thus we try to interpret the stochastic integral of a process H with respect to Y in a Riemann-Stieltjes sense

$$\int_s^t H_u \mathrm{d}Y_u = \lim_{n\to\infty} \sum_{\pi_n[s,t]} H_{t_k^*}(Y_{t_k} - Y_{t_{k-1}}),$$

where $\pi_n[s,t]$ is a partition of $[s,t]$ and t_k^* is a point of the interval $[t_k, t_{k+1}]$. By construction, Itô integrals are defined as an L^2-limit of simple integrals. Recall also that convergence in L^2 implies convergence in probability. Thus the Itô integral is given by the limit in probability of the Riemann-Stieltjes sum. However, the convergence does not hold almost surely. Luckily, in the case of Brownian motion we may interpret integrals in a pathwise manner in certain cases by using special choice of t_k^* and partition $\pi_n[s,t]$.

Next we define stochastic integral with respect to Brownian motion in a pathwise manner. We choose the partition $\pi_n[s,t]$ to be dyadic i.e.

$$\pi_n[s,t] = \left\{ t_k = \frac{k}{2^n}(t-s) + s | k = 0, \ldots, 2^n \right\}$$

and choose $t_k^* = t_{k-1}$. With these choices we can define a pathwise forward integral as an almost sure limit of Riemann-Stieltjes sum.

Remark. The partitions are chosen to be dyadic for simplicity. The results hold for any partition, which satisfies $\pi_n \subset \pi_{n+1}$.

Definition 2.2.10 (Pathwise forward integral). *The pathwise forward integral of a process H with respect to Brownian motion W over an interval $[0,T]$ is given by the almost sure limit*

$$\int_0^T H_u \mathrm{d}W_u := \lim_{n\to\infty} \sum_{\pi_n[0,T]} H_{t_{k-1}}(W_{t_k} - W_{t_{k-1}}) \tag{2.18}$$

if the limit exists. The integral over interval $[s,t]$ *is given by*

$$\int_s^t H_u \mathrm{d}W_u := \int_0^T I_{[s,t]}(u) H_u \mathrm{d}W_u$$

if the limit exists.

Remark. Assume that the pathwise integral of H with respect to Brownian motion exists. If the condition

$$\mathbb{E}\left[\int_0^T H_s^2 \mathrm{d}s\right] < \infty$$

is satisfied, then we may also interpret the integral

$$\int_0^T H_s \mathrm{d}W_s$$

as an Itô integral too. Recall also that almost sure convergence implies convergence in probability. Thus in this case, the Itô integral and the pathwise integral are the same. Consider next a case where

$$\int_0^T H_s^2 \mathrm{d}s < \infty \quad a.s..$$

Then we can stop the process with stopping times θ_k and define the stopped Itô integral $N_t^{\theta_k}$. If we stop the pathwise integral also, we have that the stopped integrals are the same. Recall that the integral process

$$\int_0^t H_s \mathrm{d}W_s$$

is defined as the almost sure limit of the stopped integral processes. Passing to the limit, we have that the pathwise integral and the limit of Itô integrals are the same also. Thus, the Itô integral and pathwise integral are the same whenever

$$\int_0^T H_s^2 \mathrm{d}s < \infty \quad a.s..$$

Let us next discuss when the limit in (2.18) exists. We make use of the following lemma.

Lemma 2.2.11. *For a continuous function $g : [0,T] \to \mathbb{R}$ we have*

$$\int_0^T g(W_s)\mathrm{d}s = \lim_{n\to\infty} \sum_{t_k\in\pi_n[0,T]} g(W_{t_k})(W_{t_{k+1}} - W_{t_k})^2 a.s.. \tag{2.19}$$

Proof. See Revuz & Yor, 1999. □

Next we show that the Itô's formula is valid with pathwise integrals.

Theorem 2.2.12. *Assume that $F \in C^{1,2}((\mathbb{R}_+, \mathbb{R})$. Then*

$$\begin{aligned} F(T, W_T) &= F(0, W_0) + \textstyle\int_0^T F_x(s, W_s)\mathrm{d}W_s \\ &+ \tfrac{1}{2}\textstyle\int_0^T F_{xx}(s, W_s)\mathrm{d}s + \int_0^T F_t(s, W_s)\mathrm{d}s, \end{aligned} \tag{2.20}$$

where the integrals are pathwise forward integrals.

Proof. Clearly, we have

$$F(T, W_T) = F(0, W_0) + \sum_{\pi_n[0,T]} \Delta F(t_i, W_{t_i}),$$

where

$$\Delta F(t_i, W_{t_i}) = F(t_i, W_{t_i}) - F(t_{i-1}, W_{t_{i-1}}).$$

Using Taylor expansion on ΔF we obtain that

$$\begin{aligned} \Delta F(t_i, W_{t_i}) &= F_t(t_{i-1}, W_{t_{i-1}})(t_i - t_{i-1}) \\ &+ F_x(t_{i-1}, W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) \\ &+ \tfrac{1}{2}F_{xx}(t_{i-1}, W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})^2 + R_t^{i,n} + R_x^{i,n}, \end{aligned}$$

where R_t and R_x are correction terms of the form

$$R_t^{i,n} = \left[F_t(W_{t_{i-1}}, \tau_i) - F_t(W_{t_{i-1}}, t_{i-1})\right]\Delta t_i$$

for some $\tau_i \in (t_i, t_{i-1})$ and

$$R_x^{i,n} = \frac{1}{2}\left[F_{xx}(\eta_i, t_{i-1}) - F_{xx}(W_{t_{i-1}}, t_{i-1})\right](\Delta W_i)^2$$

for some $\eta_i \in (\min(W_{t_i}, W_{t_{i-1}})), \max(W_{t_i}, W_{t_{i-1}}))$. Take sums over $\pi_n[0,T]$ and let $n \to \infty$ to obtain

$$\begin{aligned} F(T,W_T) - F(0,W_0) &= \lim_{n\to\infty} \textstyle\sum_{\pi_n[0,T]} F_t(t_{i-1}, W_{t_{i-1}})(t_i - t_{i-1}) \\ &+ \lim_{n\to\infty} \textstyle\sum_{\pi_n[0,T]} F_x(t_{i-1}, W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) \\ &+ \tfrac{1}{2} \lim_{n\to\infty} \textstyle\sum_{\pi_n[0,T]} F_{xx}(t_{i-1}, W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})^2 \\ &+ . \lim_{n\to\infty} \textstyle\sum_{\pi_n[0,T]} (R_x^{i,n} + R_t^{i,n}). \end{aligned}$$

We need to show that the second limit exists as an almost sure limit. The first limit is the classical Riemann integral

$$\lim_{n\to\infty} \sum_{\pi_n[0,T]} F_t(t_{i-1}, W_{t_{i-1}})(t_i - t_{i-1}) = \int_0^T F_s(s, W_s) \mathrm{d}s$$

and the third limit converges to

$$\int_0^t F_{xx}(s, W_s) \mathrm{d}s$$

by Lemma 2.2.11. In order to complete the proof, we need to show that

$$\lim_{n\to\infty} \sum_{\pi_n[0,T]} (R_x^{i,n} + R_t^{i,n}) = 0$$

almost surely. We have

$$\begin{aligned} & \textstyle\sum_{\pi_n[0,T]} R_t^{i,n} \\ = \;& \textstyle\sum_{\pi_n[0,T]} \left[F_t(W_{t_{i-1}}, \tau_i) - F_t(W_{t_{i-1}}, t_{i-1}) \right] \Delta t_i \\ \leq \;& \sup_{|t-s| \leq t_i - t_{i-1}} |F_t(W_{t_{i-1}}, t) - F_t(W_{t_{i-1}}, s)| T. \end{aligned}$$

This goes to zero, since F_t is continuous. In similar way,

$$\lim_{n\to\infty} \sum_{\pi_n[0,T]} R_x^{i,n} = 0$$

since Brownian motion has continuous paths and F_{xx} is continuous. □

2.3 Local time and Tanaka's formula

One of the key factors in stochastic analysis is the Itô's formula (2.11). However, the formula assumes that the function is smooth enough which narrows the generality. Therefore it has been of major study to extend the Itô's formula somehow for a wider class of functions. For example, Itô's formula has been extended to consider convex functions (Revuz & Yor, 1999). Using this for a function $f(x) = (x-K)^+$ we obtain the well-known Tanaka's formula. Here $(x-K)^+$ denote the positive part of $x - K$ i.e. $\max(x-K, 0)$. Similarly, $(x-K)^-$ denote the negative part $-\min(x-K, 0)$.

Definition 2.3.1. *A function f is convex if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for any x, y and $\lambda \in (0,1)$.

Remark. Every convex function has a left-hand derivative and a right-hand derivative.

Definition 2.3.2. *a sign function $sign(x)$ is given by*

$$sign(x) = \begin{cases} 1 & , x > 0 \\ -1 & , x \leq 0 \end{cases}$$

Theorem 2.3.3. *Let f be a convex function with left-hand derivative f^* and let X be a continuous semimartingale. Then there exists a continuous increasing process A_t^f such that*

$$f(X_t) = f(X_0) + \int_0^t f^*(X_u)\mathrm{d}X_u + A_t^f. \tag{2.21}$$

Proof. See Revuz & Yor, 1999. □

Note that the result implies that the class of semimartingales is closed under convex transforms. Next we prove the well-known Tanaka's formula by using this lemma.

Corollary 2.3.4 (Tanaka). *Let X be a continuous semimartingale. Then, for any K, there exists a continuous increasing process $L_t^K(X)$ such that*

$$|X_t - K| = |X_0 - K| + \int_0^t sign(X_s - K)\mathrm{d}X_s + L_t^K(X), \tag{2.22}$$

$$(X_t - K)^+ = (X_0 - K)^+ + \int_0^t I_{X_s > K}\mathrm{d}X_s + \frac{1}{2}L_t^K(X) \tag{2.23}$$

and

$$(X_t - K)^- = (X_0 - K)^- - \int_0^t I_{X_s \le K}\mathrm{d}X_s + \frac{1}{2}L_t^K(X). \tag{2.24}$$

Proof. Clearly all three functions on left sides of equations are convex. First we observe that

$$X_t - K = (X_t - K)^+ - (X_t - K)^-$$

and

$$|X_t - K| = (X_t - K)^+ + (X_t - K)^-.$$

Thus (2.22) follows by summing (2.23) and (2.24) and we have to prove only the latter two. On both equations the integrands are left-hand derivatives of the corresponding function. Thus, by Theorem 2.3.3, there are continuous increasing processes A_1 and A_2 such that

$$(X_t - K)^+ = (X_0 - K)^+ + \int_0^t I_{X_s > K}\mathrm{d}X_s + A_1$$

and

$$(X_t - K)^- = (X_0 - K)^- - \int_0^t I_{X_s \le K}\mathrm{d}X_s + A_2.$$

Substituting these we obtain

$$X_t - X_0 = \int_0^t \mathrm{d}X + \frac{1}{2}(A_1 - A_2).$$

Hence $A_1 = A_2$ a.s. and we are done by putting

$$\frac{1}{2}L_t^K(X) = A_1.$$

□

Remark. It is clear from equations (2.22)-(2.24) that $L_t^K(X)$ starts from the origin. It is also known that the process $K \to L_t^K(X)$ is right continuous with left hand limits (Revuz & Yor, 1999).

The process $L_t^K(X)$ is called the local time of X at K up to t. Since the process $L_t^K(X)$ is increasing, we can associate it with a random measure $\mathrm{d}L_t^K(X)$ (Revuz & Yor, 1999).

Theorem 2.3.5. *Let X be a continuous semimartingale and $L_t^K(X)$ its local time. Then the measure $\mathrm{d}L_t^K(X)$ is carried by the set $\{t : X_t = K\}$ almost surely i.e.*

$$\int_0^t |X_s - K| \mathrm{d}L_s^K(X) = 0 \quad a.s.. \tag{2.25}$$

Proof. Put $Y_t = |X_t - K|$. Then we know that Y is a continuous semimartingale. Apply Itô's formula for a function $F(x) = x^2$ to obtain

$$Y_t^2 = Y_0^2 + 2\int_0^t Y_s \mathrm{d}Y_s + < Y, Y >_t$$

which is equivalently

$$\begin{aligned} (X_t - K)^2 &= (X_0 - K)^2 + 2\int_0^t |X_s - K| \mathrm{d}|X_s - K| \\ &+ < |X - K|, |X - K| >_t . \end{aligned} \tag{2.26}$$

According to (2.22), we have

$$\mathrm{d}|X_s - K| = sign(X_s - K)\mathrm{d}X_s + \mathrm{d}L_s^K(X).$$

Moreover, we have that

$$< |X - K|, |X - K| >_t = < X, X >_t .$$

Substitute these into (2.26) to obtain

$$\begin{aligned} (X_t - K)^2 &= (X_0 - K)^2 + 2\int_0^t |X_s - K| sign(X_s - K)\mathrm{d}X_s \\ &+ 2\int_0^t |X_s - K| \mathrm{d}L_s^K(X) + < X, X >_t . \end{aligned}$$

Note that $|X_s - K| sign(X_s - K) = X_s - K$. Classical Itô's formula yields

$$(X_t - K)^2 = (X_0 - K)^2 + 2\int_0^t (X_s - K)\mathrm{d}X_s + < X, X >_t .$$

Comparing these we obtain

$$\int_0^t |X_s - K| \mathrm{d}L_s^K(X) = 0$$

almost surely. □

Remark. The result means that in some sense, $L_t^K(X)$ increases only when X is at the level K.

The local time process is much more interesting than just a process occuring in Tanaka's formula, and thus it has been studied widely. Actually, the Tanaka-Meyer theorem states that for each convex function, the process A_t^f of Theorem 2.3.3 is determined by the distributional derivative of the function and the the local time $L_t^K(X)$.

Theorem 2.3.6 (Tanaka-Meyer). *Let f be a difference of two convex functions and X a continuous semimartingale. Then*

$$f(X_t) = f(X_0) + \int_0^t f^*(X_u)\mathrm{d}X_u + \frac{1}{2}\int_{\mathbb{R}} L_t^a(X) f^{\cdot\cdot}(\mathrm{d}a), \tag{2.27}$$

where f^* is the left-hand derivative of f and $f^{\cdot\cdot}$ is the second derivative of f in the sense of distributions.

Proof. See Revuz & Yor, 1999. □

Remark. Every convex function f has the second derivative $f^{\cdot\cdot}$ in the sense of distributions and it is a positive measure. Thus, by linearity, a difference of two convex function has the second derivative $f^{\cdot\cdot}$ in the sense of distributions also.

Remark. Recall that f is concave if $-f$ is convex. Thus the second derivative $f^{\cdot\cdot}$ in the sense of distributions is a negative measure for concave function f.

For further use we need also the following Occupation time formula.

Theorem 2.3.7 (Occupation time formula). *Let X be a continuous semimartingale and $L_t^K(X)$ its local time at K. Then for each positive Borel-measurable function $g(x)$*

$$\int_0^t g(X_s)\mathrm{d} < X, X >_s = \int_{\mathbb{R}} g(K) L_t^K(X) \mathrm{d}K. \tag{2.28}$$

Proof. The proof can be found for example in Revuz & Yor, 1999. □

Remark. If we put $g(x) = 1$ we obtain an equation

$$< X, X >_t = \int_{\mathbb{R}} L_t^K(X) \mathrm{d}K.$$

Remark. Let A be a measurable set and put $g(x) = I_A(x)$. Then (2.28) takes the form

$$\int_0^t I_{X_s \in A} \mathrm{d} < X, X >_s = \int_A L_t^K(X) \mathrm{d}K. \tag{2.29}$$

The right hand side is a measure which is absolutely continuous with respect to a Lebesgue measure and the corresponding Radon-Nikodym derivative is the local time $L_t^K(X)$. The left hand side is an occupation type measure, which measures the time X has spent on the set A with respect to a random clock $\mathrm{d} < X, X >$. From this fact arises the term local time. In the case of Brownian motion W, the left hand side is exactly the classical occupation measure

$$\int_0^t I_{W_s \in A} \mathrm{d}s,$$

which measures the time that the path of Brownian motion has spent on the set A.

From the Occupation time formula we can derive the local time process as an almost sure limit of a certain sequence of processes. For this we need the following one dimensional case of Lebesgue differentiation theorem.

Lemma 2.3.8 (Lebesgue differentiation theorem). *Let f be a measurable function. Then*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_x^{x+\epsilon} f(y) \mathrm{d}y = f(x) \tag{2.30}$$

for almost every point $x \in \mathbb{R}$.

Remark. If f is continuous, then (2.30) holds for every x.

Lemma 2.3.9. *Let X be a continuous semimartingale and $L_t^K(X)$ its local time at K. Then the local time can be given by the almost sure limit*

$$L_t^K(X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t I_{K \leq X_s < K+\epsilon} \mathrm{d} < X, X >_s . \tag{2.31}$$

Proof. Put

$$g(x) = \frac{1}{\epsilon} I_{[K,K+\epsilon)}(x)$$

in the Occupation time formula (2.28) to conclude that

$$\frac{1}{\epsilon} \int_0^t I_{K \leq X_s < K+\epsilon} \mathrm{d} < X, X >_s = \frac{1}{\epsilon} \int_K^{K+\epsilon} L_t^a(X) \mathrm{d}a$$

almost surely. Next we let $\epsilon \to 0$. In order to complete the proof, we need to show that the process $L_T^a(S)$ is continuous with respect to a on the interval $[K, K+\epsilon)$. Then by Lebesgue differentiation theorem 2.3.8 we conclude that (2.31) holds for every K. But this follows from the fact that the process $K \to L_t^K(X)$ is right continuous. Indeed, by right continuity

$$\lim_{a \to K+} L_t^a(S) = L_t^K(S),$$

which implies that with sufficiently small ϵ, the process $L_t^a(S)$ is continuous when $a \in [K, K+\epsilon)$. Hence (2.31) holds for every K. □

The local time $L_t^K(X)$ represents the time X has spent on the level K. Therefore we might want to say something practical about the process, such as the expected value. However, often the expected value cannot be computed or can be quite tricky. This will be discussed in chapter 4.

Example. Let W be a Brownian motion and consider the local time $L_t^K(W)$ of W at K. From the Occupation time formula and by Fubini's theorem we obtain

$$\int_0^t \mathbb{E}[g(X_s)] \mathrm{d}s = \int_{\mathbb{R}} g(K) \mathbb{E}[L_t^K(W)] \mathrm{d}K.$$

On the other hand,

$$\int_0^t \mathbb{E}[g(X_s)] \mathrm{d}s = \int_0^t \int_{\mathbb{R}} g(K) \phi_{0,s}(K) \mathrm{d}K \mathrm{d}s = \int_{\mathbb{R}} g(K) \int_0^t \phi_{0,s}(K) \mathrm{d}s \mathrm{d}K$$

where $\phi_{0,s}$ is the probability density function of the normal distribution $N(0, s)$. Since g is arbitrary, we obtain that

$$\mathbb{E}[L_t^K(W)] = \int_0^t \phi_{0,s}(K)\mathrm{d}s.$$

We have

$$\begin{aligned}
\int_0^t \phi_{0,s}(K)\mathrm{d}s &= \tfrac{1}{\sqrt{2\pi}} \int_0^t \tfrac{1}{sqrts} e^{-\frac{K^2}{2s}}\mathrm{d}s \\
&\underbrace{=}_{y=\frac{1}{\sqrt{s}}} \tfrac{1}{\sqrt{2\pi}} \int_{t^{-\frac{1}{2}}}^{\infty} y e^{-\frac{K^2y^2}{2}} \cdot 2y^{-3}\mathrm{d}y \\
&= \tfrac{2}{\sqrt{2\pi}} \int_{t^{-\frac{1}{2}}}^{\infty} y^{-2} e^{-\frac{K^2y^2}{2}}\mathrm{d}y.
\end{aligned}$$

Integration by parts formula yields

$$\int y^{-2} e^{-\frac{K^2y^2}{2}}\mathrm{d}y = -\frac{1}{y} e^{-\frac{K^2y^2}{2}} - K^2 \int e^{-\frac{K^2y^2}{2}}\mathrm{d}y.$$

Thus,

$$\mathbb{E}[L_t^K(W)] = \frac{2\sqrt{t}}{\sqrt{2\pi}} e^{-\frac{K^2}{2t}} - \frac{2K^2}{\sqrt{2\pi}} \int_{t^{-\frac{1}{2}}}^{\infty} e^{-\frac{K^2y^2}{2}}\mathrm{d}y.$$

The last integral can be computed in terms of the cumulative distribution function $\Phi_{a,b}$ of normal distribution with parameters a and b. Indeed, with $c = 1/K$ we have

$$\begin{aligned}
\tfrac{2K^2}{\sqrt{2\pi}} \int_{t^{-\frac{1}{2}}}^{\infty} e^{-\frac{K^2y^2}{2}}\mathrm{d}y &= \tfrac{2}{c\sqrt{c}} \int_{t^{-\frac{1}{2}}}^{\infty} \tfrac{1}{\sqrt{2\pi c}} e^{-\frac{y^2}{2c}}\mathrm{d}y \\
&= \tfrac{2}{c\sqrt{c}} \left[1 - \Phi_{0,c}(\tfrac{1}{\sqrt{t}})\right].
\end{aligned}$$

Thus we have

$$\mathbb{E}[L_t^K(W)] = \frac{\sqrt{2t}}{\sqrt{\pi}} e^{-\frac{K^2}{2t}} - 2K\sqrt{K}\left[1 - \Phi(\frac{\sqrt{K}}{\sqrt{t}})\right],$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution with parameters 0 and 1. With $K = 0$, this equals

$$\mathbb{E}[L_t^0(W)] = \frac{\sqrt{2t}}{\sqrt{\pi}}.$$

Chapter 3

Black and Scholes model

In this chapter we introduce the Black and Scholes model. Section 3.1 is about modelling the finance and contains mathematical definitions of economical consepts such as arbitrage opportunity and self-financing portfolio. In section 3.2, we introduce the Black and Scholes market model. We prove the basic properties of the model such as completeness and arbitrage-free property. Section 3.3 concentrates on European type options. We find the hedge of a European option and derive the value of a portfolio precisely. Specially, we derive the hedge of a European call.

3.1 Modelling the finance

The global market system is a complex system. For example, the price of a product depends on demand and supply. However, the demand and supply can depend on many things such as the current political situation, time of the year, if there are competitive products on the market, technological issues and enviromental issues. Moreover, many of these factors can change rapidly and are hard to predict. Therefore there have to be simplified models in order to analyze and predict the behaviour of the market system.

One of the major issues in finance is to analyze options. A usual option is based on the price of some product such as stock. Therefore a typical model is based on some assumptions of how the stock prices behave. The model must fit reality as well as possible. Thus we need the model to be arbitrage-free. We also want that the hedging is possible in some sense. Typically, we have to narrow the class of possible portfolios somehow.

In order to construct a model that fits reality, we first need to define economical concepts such as arbitrage mathematically. We consider a model with a bank account and one stock. Let S_t denote the price of the stock at time t and B_t the cash amount on bank acount at time t.

Definition 3.1.1. *A trading strategy π is a predictable*[1] *process $\pi_t = (\beta_t, \gamma_t)$ where β_t represents the money amount on bank account at t and γ_t the investment on stocks at t.*

Remark. The strategy π_t at the moment t is also called a portfolio or position at t.

The assumption that our strategies π are predictable is a realistic assumption, hence the investor has to choose her own position. Even predictable strategies do not always serve the purpose of the model and we often have to narrow the class of allowed strategies. This will be considered in the next section.

Definition 3.1.2. *Value of portfolio π at t is given by*

$$V_t = \beta_t B_t + \gamma_t S_t. \tag{3.1}$$

Since we work in a continuous time model, we use a continuous interest given by e^{rt}. Thus, B_t is given by

$$B_t = e^{rt} B_0.$$

We denote the discounted stock price with $\overline{S}_t$ and discounted value process with $\overline{V}_t$. Formally:

$$\begin{aligned}\overline{S}_t &= e^{-rt} S_t \\ \overline{V}_t &= e^{-rt} V_t.\end{aligned}$$

Usually we are able to change our position π dynamically in time. However, we do not want to change the position in a way which requires more invested capital. Thus we want that our strategy is self-financing. Consider that we invest our initial capital V_0 into position (β_0, γ_0) and have a chance to change our position at time t.

[1]Economically it is more reasonable to assume that our strategies are adapted. This means that as we know the prices, we also know what the best strategy should be. However, Brownian motion has continuous paths and thus the integral with respect to Brownian motion does not observe whether the integrand is predictable or adapted.

The value of our portfolio at the beginning is

$$V_0 = \beta_0 B_0 + \gamma_0 S_0$$

and at time t

$$V_t = \beta_0 B_t + \gamma_0 S_t.$$

Now we change our position and invest this capital into portfolio (β_t, γ_t). Since the capital is the same, we have

$$\beta_t B_t + \gamma_t S_t = \beta_0 B_t + \gamma_0 S_t.$$

Thus we obtain

$$\begin{aligned} \Delta V_t &= V_t - V_0 \\ &= \beta_t B_t + \gamma_t S_t - \beta_0 B_0 - \gamma_0 S_0 \\ &= \beta_0 \Delta B_t + \gamma_0 \Delta S_t. \end{aligned}$$

This deduction justifies the following definition.

Definition 3.1.3. *A strategy π is self-financing if*

$$\mathrm{d}V_t = \beta_t \mathrm{d}B_t + \gamma_t \mathrm{d}S_t. \tag{3.2}$$

Remark. This definition is heuristic, we do not assume anything from the process S and thus we do not know whether the integral

$$\int_0^t \gamma_u \mathrm{d}S_u$$

exists. If S is geometric Brownian motion, then the integral can be defined as in Chapter 2. This is the case in Black and Scholes model.

An arbitrage opportunity is a way of making profit without risk. Mathematically this means that one can make profit with positive probability and the probability of losing money is zero.

Definition 3.1.4. *A strategy $\pi_t = (\beta_t, \gamma_t)$ is an arbitrage opportunity if its value process satisfies $V_0 \leq 0$, $V_t \geq 0$ a.s. and $\mathbb{P}(V_t > 0) > 0$.*

Definition 3.1.5. *A claim is a random variable F such that $0 \leq F < \infty$.*

Remark. Usually we assume that the claim is $\mathcal{F}_T$-measurable. This is realistic, since T represents the expiration date of an option and $\mathcal{F}_T$ represents the information, how the prices have behaved up to T.

Definition 3.1.6. *An $\mathcal{F}_t$-measurable claim F can be hedged, if there is a portfolio $\pi_t = (\beta_t, \gamma_t)$ such that*

$$F = \beta_t B_t + \gamma_t S_t$$

almost surely. Such a portfolio is called replicating portfolio.

Remark. Usually we work with self-financing strategies and thus the only interesting part is γ_t. The cash amount on bank account β_t follows from the self-financing condition. We call γ_t the hedge of the claim.

Remark. The model, where every claim can be hedged is a complete model.

One of the key issues in finance is the question of how to price claims correctly. The correct price should be fair to both sides of the bargain. For example, consider a European call option, which includes risk for the seller. Thus the seller should receive money, otherwise she should not agree to the contract. On the other hand, if the price of the option is too high, then the buyer should not agree. The equilibrium price is called the fair price of the option. If the claim can be hedged, then the value of the claim is the same as the value of the replicating portfolio. Thus, the price of an option should be the value of the replicating portfolio.

Definition 3.1.7. *Let the market be arbitrage-free and assume that the $\mathcal{F}_T$-measurable claim F can be hedged. The fair price $c_t(F)$ of the claim at the moment t is the value of the replicating portfolio i.e.*

$$c_t(F) = V_t.$$

Definition 3.1.8. *A measure Q is an equivalent martingale measure if it is equivalent to the original measure $\mathbb{P}$ and the discounted stock price $\overline{S}_t$ is a Q-martingale.*

In order to show that the model is arbitrage-free one needs to check that the condition of definition 3.1.4 does not hold for any strategy. Clearly, this can be complicated. Luckily the absence of arbitrage is related to the existence of equivalent martingale measures. The purpose of the martingale measure is to make the discounted value process a martingale. Intuitively speaking, arbitrage opportunities vanish, since the expected present value of every self-financing porfolio equals the initial capital.

3.2 The model

The Black and Scholes model with one stock and bond is a continuous time model, where the stock price follows geometric Brownian motion. Formally, the model is given by the following definition.

Definition 3.2.1. *The Black and Scholes model involves a Wiener space (Ω, $\mathcal{F}$,$\mathbb{F}$,$\mathbb{P}$) with bond and stock price processes. The bond price B follows the dynamics*

$$\mathrm{d}B_t = r\mathrm{d}t, B_0 = 1 \tag{3.3}$$

and the stock price S follows geometric Brownian motion i.e.

$$\frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t, S_0 = S_0. \tag{3.4}$$

The model involves three parameters, r, μ and σ. Here r represents the interest rate of a bank account, μ is the expected profit rate one has on the stock and σ is the volatility of the stock. The volatility represents the growing rate of the quadratic variation and measures the fluctuation of the price.

The first differential equation (3.3) is deterministic and we know that it has a solution

$$B_t = e^{rt}.$$

The solution to stochastic differential equation (3.4) is geometric Brownian motion and is given by

$$S_t = S_0 e^{\mu t + \sigma W_t - \frac{1}{2}\sigma^2 t}.$$

Let us next discuss the self-financing condition in the Black and Scholes model. Substituting (3.3) and (3.4) in (3.2) we get

$$\mathrm{d}V_t = \beta_t B_t r\mathrm{d}t + \gamma_t \mu S_t \mathrm{d}t + \gamma_t \sigma S_t \mathrm{d}W_t.$$

For discounted stock price $\overline{S}_t = \frac{S_t}{B_t}$ we have

$$\begin{aligned} \mathrm{d}\overline{S}_t &= \tfrac{1}{B_t}\mathrm{d}S_t + S_t \mathrm{d}\tfrac{1}{B_t} \\ &= \tfrac{1}{B_t}\mathrm{d}S_t - r\overline{S}_t \mathrm{d}t \\ &= \sigma \overline{S}_t \mathrm{d}W_t + \mu \overline{S}_t \mathrm{d}t - r\overline{S}_t \mathrm{d}t. \end{aligned}$$

Using this we get that the discounted value process $\overline{V}$ follows the dynamics

$$\begin{aligned} \mathrm{d}\overline{V}_t &= \mathrm{d}\tfrac{V_t}{B_t} \\ &= \tfrac{1}{B_t}\mathrm{d}V_t + V_t \mathrm{d}\tfrac{1}{B_t} \\ &= \beta_t r \mathrm{d}t + \gamma_t \mu \overline{S}_t \mathrm{d}t \\ &+ \gamma_t \sigma \overline{S}_t \mathrm{d}W_t - V_t r \tfrac{1}{B_t}\mathrm{d}t \\ &= \gamma_t \mathrm{d}\overline{S}_t. \end{aligned}$$

Thus,

$$\overline{V}_t = \overline{V}_0 + \int_0^t \gamma_u \mathrm{d}\overline{S}_u. \tag{3.5}$$

We use this notion to prove the arbitrage-free property of the model.

3.3 Hedging and no-arbitrage pricing

In this section we show that if we narrow the class of allowed strategies, the Black and Scholes model is a complete, arbitrage-free market model.

Definition 3.3.1. *A self-financing strategy $\pi_t = (\beta, \gamma)$ is admissible, if there exists a positive constant C such that the value function satisfies*

$$V_t \geq -C \quad \forall t \in [0, T].$$

Remark. This condition is economically reasonable. It simply means that there is a limit to how much one can loose on her investment.

Lemma 3.3.2. *Assume that a strategy π_t is admissible. Then the stochastic integral* *process*

$$N_t = \int_0^t \sigma\gamma_u S_u \mathrm{d}W_u$$

is a supermartingale.

Proof. The proof is based on Fatou's lemma. In order to apply Fatou's lemma we need that the sequence has a negative integrable lower bound. The condition $V_t \geq -C$ implies this.

Now we know that the process N_t is a local martingale. Define a stopping time θ_k by

$$\theta_k = \inf\left\{t : \int_0^t \sigma^2\gamma_u^2 S_u^2 \mathrm{d}s \geq k\right\}.$$

Then the stopped process $N_t^{\theta_k}$ is a martingale and θ_k tends to infinity as k tends to infinity. Thus the sequence $(\theta_k)_{k\geq 1}$ is a localising sequence. On the other hand, the integral process N_t is given by the almost sure limit of the processes $N_t^{\theta_k}$. By Fatou's lemma we have

$$\begin{aligned}
\mathbb{E}[N_t|\mathcal{F}_s] &= \mathbb{E}[\liminf N_t^{\theta_k}|\mathcal{F}_s] \\
&\leq \liminf \mathbb{E}[N_t^{\theta_k}|\mathcal{F}_s] \\
&= \liminf N_s^{\theta_k} \\
&= N_s.
\end{aligned}$$

Thus we have the result. □

Remark. We also have that the stochastic integral

$$\int_0^t \sigma\gamma_u \overline{S}_u \mathrm{d}W_u$$

is a supermartingale.

This supermartingale-property implies the arbitrage-free condition of the model justified in next theorem.

Theorem 3.3.3. *Assume that π is an admissible strategy. Then the strategy allows no arbitrage.*

Proof. As usual, the existence of an equivalent martingale measure implies the arbitrage-free property of the model. Indeed, we construct the equivalent martingale measure Q such that the discounted price is an exponential martingale. If we have this, equation (3.5) and Lemma 3.3.2 implies that the discounted value process is also a supermartingale. Finally, we show that arbitrage cannot exists when the discounted value process is supermartingale.

Direct computation gives

$$\begin{aligned}
\mathrm{d}\overline{S}_t &= \tfrac{1}{B_t}\mathrm{d}S_t + S_t\mathrm{d}\tfrac{1}{B_t} \\
&= e^{-rt}\mathrm{d}S_t - rS_te^{-rt}\mathrm{d}t \\
&= e^{-rt}(\mathrm{d}S_t - rS_t\mathrm{d}t) \\
&= e^{-rt}(\mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t - rS_t\mathrm{d}t) \\
&= (\mu - r)\overline{S}_t\mathrm{d}t + \sigma\overline{S}_t\mathrm{d}W_t.
\end{aligned}$$

Thus the discounted stock price $\overline{S}$ follows the dynamics

$$\frac{\mathrm{d}\overline{S}_t}{\overline{S}_t} = (\mu - r)\mathrm{d}t + \sigma\mathrm{d}W_t.$$

Define a process W_t^Q by

$$W_t^Q = W_t + \frac{\mu - r}{\sigma}t.$$

Then

$$\frac{\mathrm{d}\overline{S}_t}{\overline{S}_t} = \sigma\mathrm{d}W_t^Q.$$

Next we define the measure Q such that W_t^Q is Brownian motion. Let Q be a measure absolutely continuous with respect to $\mathbb{P}$ such that the corresponding Radon-Nikodym derivative is given by

$$Z = e^{-\frac{\mu-r}{\sigma}W_T - \left(\frac{\mu-r}{\sigma}\right)^2 T}.$$

Then by Girsanov theorem 2.2.8 Q is equivalent to $\mathbb{P}$, W^Q is Brownian motion with respect to the measure Q and thus the process $\overline{S}_t$ is an exponential martingale with respect to the measure Q. Hence we have found an equivalent martingale measure Q such that the discounted price process is a Q-martingale. Now equation (3.5) takes the form

$$\overline{V}_t = \overline{V}_0 + \int_0^t \sigma\gamma_u \overline{S}_u \mathrm{d}W_u^Q.$$

Moreover, the integral is a supermartingale by Lemma 3.3.2 and thus $\overline{V}_t$ is a supermartingale also. Recall the Definition 3.1.4 of an arbitrage opportunity. Since Q is equivalent to $\mathbb{P}$, the same conditions hold with respect to measure Q also and with the discounted value process. Let π be an admissible arbitrage opportunity. Since now the corresponding discounted value process $\overline{V}_t$ is a supermartingale with respect to Q, we have

$$\mathbb{E}_Q[\overline{V}_t] \leq \mathbb{E}_Q[\overline{V}_0].$$

On the other hand, $\mathbb{E}_Q[\overline{V}_0] \leq 0$ and $\mathbb{E}_Q[\overline{V}_t] > 0$. This is a contradiction. Thus, π cannot be an arbitrage opportunity. □

Remark. The assumption that the strategy is admissible is not in vain. Indeed, one can construct an arbitrage opportunity π which does not satisfy the condition in Definition 3.3.1.

Remark. If we assume that

$$\mathbb{E}\left[\int_0^T \gamma_t^2 S_t^2 \mathrm{d}t\right] < \infty,$$

we have that the stochastic integral is a martingale instead of supermartingale and we have the same result. This implies that the value process is square integrable and in this case, results hold. However, there is no economical interpreptation for this.

Remark. Under the measure Q, the discounted price process $\overline{S}$ is an exponential martingale. The same results follows, if we only assume that the drift μ equals the risk-free interest rate r. Thus we can analyse the model under measure Q simply by putting $\mu = r$. The economical interpretation of this is that the expected profit of stock equals the profit of a bank account.

Now we know that the Black and Scholes model is arbitrage-free with admissible strategies and the result follows from Girsanov theorem 2.2.8. In similar way, it turns out that the model is complete in a sense that every Q-square integrable claim can be replicated. This follows from Itô-Clark representation theorem 2.2.9.

Theorem 3.3.4. *Every $\mathcal{F}_T$-measurable claim $F \in L^2(Q)$ can be replicated.*

Proof. Without loss of generality, we may assume $r = 0$. Then $\overline{V}_t = V_t$ and $\overline{S}_t = S_t$. We need to find a strategy $\pi_t = (\beta_t, \gamma_t)$ such that

$$F = V_0 + \int_0^T \gamma_t \mathrm{d}S_t$$

or equivalently

$$F = V_0 + \int_0^T \gamma_t \sigma S_t \mathrm{d}W_t^Q.$$

From Itô-Clark representation theorem 2.2.9 we know that there exists a unique square integrable, predictable process H such that

$$F = \mathbb{E}_Q[F] + \int_0^T H_t \mathrm{d}W_t^Q.$$

Put

$$\gamma_t = \frac{H_t}{\sigma S_t}.$$

Since H is predictable, so is γ_t. In order to complete the proof, we need to show that such a portfolio does not allow arbitrage. Then we have our replicating portfolio with initial capital $\mathbb{E}_Q[F]$. Recall that we only need to show that

$$\mathbb{E}_Q\left[\int_0^T H_t^2 \mathrm{d}t\right] < \infty.$$

Using Itô isometry we conclude that

$$\mathbb{E}_Q\left[\int_0^T H_t^2 \mathrm{d}t\right] = \mathbb{E}_Q\left[\int_0^T H_t^2 \mathrm{d}W_t^Q\right].$$

But F is square integrable by assumption and

$$\int_0^T H_t^2 \mathrm{d}W_t^Q = F - V_0.$$

Thus we have the result. □

Remark. We assumed that the claim is $\mathcal{F}_T$-measurable. This simply means that we know the value of the claim at the moment T.

In a discrete time setup, the completeness of the model is equivalent to the fact that the measure Q is unique (Lamberton & Lapeyre, 1996). This is true also in the Black and Scholes model.

Corollary 3.3.5. *The equivalent martingale measure is unique.*

Proof. Assume that Q and Q^* are equivalent martingale measures. Let $A \in \mathcal{F}_T$ be arbitrary. By Theorem 2.2.9 we have

$$I_A = Q(A) + \int_0^T \frac{H_t}{\sigma S_t} \mathrm{d}S_t.$$

By definition of the measure, the stochastic integral above is a Q^*-martingale. Taking expectation with respect to Q^* yields

$$Q^*(A) = Q(A)$$

and we are done. □

Now we know that every claim can be replicated. Thus the fair price of the claim at t should be the value of the replicating portfolio at t.

Corollary 3.3.6. *The fair price of an $\mathcal{F}_T$-measurable claim $F \in L^2(Q)$ at the moment t is given by*

$$c_t(F) = e^{-r(T-t)}\mathbb{E}_Q[F|\mathcal{F}_t]. \tag{3.6}$$

Proof. Let V_t be the value of the replicating portfolio. Thus, by definition, $F = V_T$ and the fair price of the claim is given by

$$c_t(F) = V_t.$$

Recall that the discounted value $\overline{V}$ is a Q-martingale. Thus

$$\overline{V}_t = \mathbb{E}_Q[\overline{V}_T|\mathcal{F}_t]$$

and by discounting we obtain

$$e^{-rt}V_t = e^{-rT}\mathbb{E}_Q[V_T|\mathcal{F}_t].$$

This implies

$$V_t = e^{-r(T-t)}\mathbb{E}_Q[F|\mathcal{F}_t]$$

and we have the result. □

3.4 European options

In last section, we showed that every measurable square integrable claim can be replicated. However, in general we do not have any practical way to compute the hedging portfolio. In this section, we consider a European type claim $f(S_T)$ and derive the hedging precisely. By Corollary 3.3.6 we have

$$c_t(f(S_T)) = V_t = e^{-r(T-t)}\mathbb{E}_Q[f(S_T)|\mathcal{F}_t], \tag{3.7}$$

where V_t is the value process of the replicating portfolio. We find a value function $v(t,x)$ such that

$$c_t(f(S_T)) = V_t = v(t, S_t).$$

Recall that computing in measure Q is the same as we put $\mu = r$ and compute in measure $\mathbb{P}$. We have, by this and direct computation,

$$\begin{aligned}
\mathbb{E}_Q[f(S_T)|\mathcal{F}_t] &= \mathbb{E}_Q[f(S_0 e^{\mu T + \sigma W_T - \frac{\sigma^2}{2}T})|\mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{P}}[f(S_0 e^{rT + \sigma W_T - \frac{\sigma^2}{2}T})|\mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{P}}[f(S_t e^{r(T-t) + \sigma(W_T - W_t) - \frac{\sigma^2}{2}(T-t)})|\mathcal{F}_t].
\end{aligned}$$

Next we make use of (2.4). Since $W_T - W_t$ is independent of $\mathcal{F}_t$, we obtain

$$\mathbb{E}_{\mathbb{P}}[f(S_t e^{r(T-t)+\sigma(W_T-W_t)-\frac{\sigma^2}{2}(T-t)})|\mathcal{F}_t] = \mathbb{E}_{\mathbb{P}}[f(x e^{r(T-t)+\sigma(W_T-W_t)-\frac{\sigma^2}{2}(T-t)})]|_{x=S_t}.$$

Since $W_T - W_t \sim N(0, T-t)$, we can substitute $W_T - W_t$ with $\sqrt{T-t}Z$, where Z follows the standard normal distribution $N(0,1)$. We get

$$\mathbb{E}_{\mathbb{P}}[f(x e^{r(T-t)+\sigma\sqrt{T-t}Z-\frac{\sigma^2}{2}(T-t)})]|_{x=S_t} = \int_{\mathbb{R}} f(S_t e^{r(T-t)+\sigma\sqrt{T-t}y-\frac{\sigma^2}{2}(T-t)})\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\mathrm{d}y.$$

Hence we have obtained a function

$$v(t,x) = e^{-r(T-t)} \int_{\mathbb{R}} f(xe^{r(T-t)+\sigma\sqrt{T-t}y-\frac{\sigma^2}{2}(T-t)}) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \mathrm{d}y. \tag{3.8}$$

Put $\theta = T - t$ and

$$z = xe^{r\theta+\sigma\sqrt{\theta}y-\frac{\sigma^2}{2}\theta}.$$

Then we can represent (3.8) in the form

$$v(t,x) = \int_{\mathbb{R}_+} f(z)k(z,x,\theta)\mathrm{d}z,$$

where the kernel k is given by

$$k(z,x,\theta) = \frac{e^{-r\theta}}{\sigma z\sqrt{2\pi\theta}} e^{-\frac{1}{2\sigma^2\theta}\left[\log\frac{z}{x}-\left(r-\frac{\sigma^2}{2}\right)\theta\right]^2}.$$

In computations we used the result (2.4). Therefore the function f has to be bounded enough so that the Fubini's theorem can be applied. It turns out that the Fubini's theorem may be applied because our claim f is square integrable. Technical details can be found in Appendix B.

Now we have found the value function $v(x,t)$ given by (3.7) for European type options, and it is of the form (3.8). Next we find the replicating portfolio explicitly for such options. We use Itô's formula (2.11), and for this the value function (3.8) has to be smooth enough. Clearly, the kernel k is smooth with respect to its parameters. Therefore, if f is bounded enough such that we can change the order of integration and derivation, we see that the value function v is smooth also. This is also true for square integrable claims $f(S_T)$. Again, technical details can be found in Appendix B. Next we assume that the value function is in $C^{1,2}([0,T),\mathbb{R})$ and find the hedge in this case.

Theorem 3.4.1. *Let $f(S_T)$ be a European option such that the corresponding value function given by (3.8) is in $C^{1,2}([0,T),\mathbb{R})$. Then the replicating portfolio $\pi_t = (\beta_t,\gamma_t)$ is given by*

$$\beta_t = e^{-rt}\left[v(t,S_t) - v_x(t,S_t)S_t\right], \tag{3.9}$$

$$\gamma_t = v_x(t,S_t). \tag{3.10}$$

Proof. If γ_t is given by (3.10), we can compute β_t from (3.1) and we get (3.9). Therefore we only need to prove equation (3.10). Let $\overline{v}(t,x)$ denote the discounted value function $\overline{v}(t,x) = e^{-rt}v(t,x)$. Clearly, $\overline{v}(t,x) \in C^{1,2}([0,T),\mathbb{R})$ whenever $v(t,x) \in C^{1,2}([0,T),\mathbb{R})$. Itô's formula (2.11) yields

$$\mathrm{d}\overline{V}_t = \overline{v}_x(t,S_t)\mathrm{d}S_t + \left[\overline{v}_t(t,S_t) + \frac{\sigma^2}{2}S_t^2\overline{v}_{xx}(t,S_t)\right]\mathrm{d}t. \tag{3.11}$$

For stock price we have

$$\mathrm{d}S_t = B_t\mathrm{d}\overline{S}_t + rS_t\mathrm{d}t.$$

Combining these we obtain

$$\mathrm{d}\overline{V}_t = \overline{v}_x(t,S_t)B_t\mathrm{d}\overline{S}_t + \left[\overline{v}_x(t,S_t)rS_t + \overline{v}_t(t,S_t) + \frac{\sigma^2}{2}S_t^2\overline{v}_{xx}(t,S_t)\right]\mathrm{d}t. \tag{3.12}$$

Now $\overline{V}$ is a Q-martingale and $\overline{S}$ is a Q-martingale. Thus the drift

$$\left[\overline{v}_x(t,S_t)rS_t + \overline{v}_t(t,S_t) + \frac{\sigma^2}{2}S_t^2\overline{v}_{xx}(t,S_t)\right]\mathrm{d}t$$

must disappear. Indeed, we have that

$$\int_0^T \overline{v}_x(t,S_t)B_t\mathrm{d}\overline{S}_t$$

is at least a local martingale with respect to Q. Thus, from equation (3.12) we get that the process

$$H_T = \int_0^T \overline{v}_x(t,S_t)rS_t + \overline{v}_t(t,S_t) + \frac{\sigma^2}{2}S_t^2\overline{v}_{xx}(t,S_t)\mathrm{d}t$$

is also a local martingale. But this process is a continuous process and has bounded variation. Therefore, by Corollary 2.2.5,

$$H_t = 0$$

identically and equation (3.12) simplifies into form

$$\mathrm{d}\overline{V}_t = \overline{v}_x(t,S_t)B_t\mathrm{d}\overline{S}_t.$$

By direct computation

$$\overline{v}_x(t, S_t)B_t = v_x(t, S_t).$$

On the other hand, by (3.5) we have

$$\mathrm{d}\overline{V}_t = \gamma_t \mathrm{d}\overline{S}_t$$

and the result follows. □

Theorem 3.4.1 gives us a direct way to compute the hedge of an option $f(S_T)$, if we know the value function and it is smooth enough. However, computing integral (3.8) can be tricky and often requires numerical methods. Next we find another way to compute the hedge of the option.

Proposition 3.4.2. *Let a European option $f(S_T)$ be square integrable with respect to Q. Moreover, assume that the corresponding value function of the replicating portfolio $v(t, x) \in C^{1,2}([0, T), \mathbb{R})$. Then $v(x, t)$ satisfies the differential equation*

$$\begin{aligned} v_t + \tfrac{\sigma^2}{2} x^2 v_{xx} + r x v_x - r v &= 0 \\ v(T, \cdot) &= f. \end{aligned} \tag{3.13}$$

Proof. By definition of the hedge, the boundary condition is satisfied. By direct computation we have

$$\mathrm{d}V_t = \mathrm{d}(B_t \overline{V}_t) = B_t \mathrm{d}\overline{V}_t + r V_t \mathrm{d}t$$

and

$$\mathrm{d}\overline{S}_t = \mathrm{d}\frac{S_t}{B_t} = \frac{1}{B_t}\mathrm{d}S_t - \frac{r S_t}{B_t}\mathrm{d}t.$$

On the other hand,

$$\mathrm{d}\overline{V}_t = v_x(t, S_t)\mathrm{d}\overline{S}_t.$$

Combine these to obtain

$$\mathrm{d}V_t = v_x(t, S_t)\mathrm{d}S_t + [r v(t, S_t) - v_x(t, S_t) r S_t]\mathrm{d}t.$$

From Itô's formula (2.11)

$$\mathrm{d}V_t = v_x(t,S_t)\mathrm{d}S_t + \left[v_t(t,S_t) + \frac{\sigma^2}{2}S_t^2 v_{xx}(t,S_t)\right]\mathrm{d}t.$$

Combining these we obtain a deterministic differential equation

$$\begin{aligned} v_t + \tfrac{\sigma^2}{2}x^2 v_{xx} + rxv_x - rv &= 0 \\ v(T,\cdot) &= f \end{aligned}$$

and we are done. □

This theorem states that our value function $v(x,t)$ satisfies the differential equation (3.13). More interesting is that the opposite fact is also true. If a function v satisfies equation (3.13), then v_x is the hedge for the claim $f(S_T)$.

Proposition 3.4.3. *Assume that a function v satisfies differential equation (3.13). Then v is the value function of the replicating portfolio of an option $f(S_T)$. Thus, v_x is the hedge for the claim $f(S_T)$.*

Proof. By Itô's formula (2.11) and equation (3.13) we have

$$\mathrm{d}V_t = [rv(t,S_t) - rS_t v_x(t,S_t)]\mathrm{d}t + v_x(t,S_t)\mathrm{d}S_t.$$

Substitute

$$\mathrm{d}S_t = B_t\mathrm{d}\overline{S}_t + rS_t\mathrm{d}t$$

to obtain

$$\mathrm{d}V_t = rv(t,S_t)\mathrm{d}t + B_t v_x(t,S_t)\mathrm{d}\overline{S}_t.$$

On the other hand,

$$\mathrm{d}V_t = B_t\mathrm{d}\overline{V}_t + rv(t,S_t)\mathrm{d}t$$

which gives

$$\mathrm{d}\overline{V}_t = v_x(t,S_t)\mathrm{d}\overline{S}_t.$$

Combining with (3.5) gives the result. □

Now we have two ways to compute the hedge, through the differential equation (3.13) and through integral (3.8) and Theorem 3.4.1. Note that both these ways result into the same hedge and the hedge does not depend on the drift μ. Economically this means that despite the fact that people may have different opinions on the expectations on the profit of the stock, the hedge and the price should be the same.

3.4.1 European call

Let us derive the hedge for the European call $f(S_T) = (S_T - K)^+$. First we need to calculate the value process (3.8). By direct computation we obtain that the fair price of a European call is given by the value function

$$v(t, S_t) = S_t \Phi\left(d_1(t, S_t)\right) - K e^{-r(T-t)} \Phi\left(d_2(t, S_t)\right) \tag{3.14}$$

where Φ is the cumulative distribution function of the standard normal distribution $N(0,1)$ and

$$\begin{aligned} d_1(t, S_t) &:= \frac{\log \frac{S_t}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2(t, S_t) &:= d_1(t, S_t) - \sigma\sqrt{T-t}. \end{aligned}$$

Computation can be found in Appendix C. Clearly,

$$\frac{\partial d_2}{\partial x}(t, x) = \frac{\partial d_1}{\partial x}(t, x).$$

It is easy to check that also

$$\frac{\partial}{\partial x}\Phi\left(d_2(t, x)\right) = \frac{x e^{r(T-t)}}{K}\frac{\partial}{\partial x}\Phi\left(d_1(t, x)\right).$$

Thus we obtain

$$\begin{aligned} \frac{\partial v}{\partial x}(t, x) &= \frac{\partial}{\partial x}\left[x\Phi\left(d_1(t, x)\right) - K e^{-r(T-t)}\Phi\left(d_2(t, x)\right)\right] \\ &= \Phi\left(d_1(t, x)\right). \end{aligned}$$

Hence the hedge is given by

$$\gamma_t = \Phi(d_1(t, S_t)).$$

Assume that the model is discounted i.e. $r = 0$. Then $\overline{V}_t = V_t$ and $\overline{S}_t = S_t$. Thus (3.5) takes the form

$$V_t = V_0 + \int_0^t \gamma_u \mathrm{d}S_u.$$

If we choose the portfolio to be the replicating portfolio, we obtain a relation

$$(S_T - K)^+ = V_0 + \int_0^T \Phi\left(d_1(t, S_t)\right) \mathrm{d}S_t, \tag{3.15}$$

where

$$V_0 = v(0, S_0) = S_0 \Phi\left(d_1(0, S_0)\right) - K\Phi\left(d_2(0, S_0)\right)$$

and

$$\begin{aligned} d_1(t, S_t) &:= \frac{\log \frac{S_t}{K} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\ d_2(t, S_t) &:= d_1(t, S_t) - \sigma\sqrt{T-t}. \end{aligned}$$

Remark. As the maturity of an option increases, the risk involved increases also and thus the price increases also. Mathematically this is same as the value function should be increasing in T. This is indeed so and can be proved by direct computation (Dana & Jeanblanc, 2003). In the next section we show this by other arguments. We also show that this is true for every convex European option.

Chapter 4

Local time and option prices

In this chapter we introduce the results. First, in section 4.1, we derive a new integral representation for the local time of geometric Brownian motion. In section 4.2 we compute the expectation of the local time through Occupation time formula and by representation derived in section 4.1. In section 4.3 we consider some applications. In section 4.4, we consider European options which satisfies the assumption of the Tanaka-Meyer theorem.

4.1 Integral representation for local time of geometric Brownian motion

Our purpose is to find a new integral representation for local time of geometric Brownian motion. This is done by comparing the Tanaka's formula and the results of section 3.4.

Theorem 4.1.1. *Let S be geometric Brownian motion, $L_T^K(S)$ its local time at K and Φ the cumulative distribution function of a standard normal distribution $N(0,1)$. Then the local time admits a representation*

$$L_T^K(S) = 2\int_0^T \left[\Phi(d_1^0(t,S_t)) - I_{S_t>K}\right] \mathrm{d}S_t + 2V_0^{T,0} - 2(S_0-K)^+ \qquad (4.1)$$

where $d_1^r(t, S_t)$ and $d_2^r(t, S_t)$ are given by

$$\begin{aligned} d_1^r(t, S_t) &= \frac{\log \frac{S_t}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2^r(t, S_t) &= d_1^r(t, S_t) - \sigma\sqrt{T-t} \end{aligned}$$

and $V_0^{T,r}$ by

$$V_0^{T,r} = S_0 \Phi\left(d_1^r(0, S_0)\right) - K e^{-rT} \Phi\left(d_2^r(0, S_0)\right) \tag{4.2}$$

i.e. the price of the European call in a model which interest rate is given by r.

Remark. From now on, a short notation V_0^T denotes the price $V_0^{T,0}$. Similarly, short notations d_1 and d_2 denote d_1^0 and d_2^0.

Proof. Now we work in a discounted model i.e. $r = 0$. According to (3.15), the European call $(S_T - K)^+$ has a representation

$$(S_T - K)^+ = V_0^T + \int_0^T \Phi\left(d_1(t, S_t)\right) \mathrm{d}S_t.$$

On the other hand, geometric Brownian motion is a semimartingale and by Tanaka's formula

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T I_{S_t > K} \mathrm{d}S_t + \frac{1}{2} L_T^K(S).$$

The right-hand sides are the same and (4.1) follows. □

Let us check that the local time starts from the origin as it should. If $T = 0$, then the stochastic integral equals zero. Thus we need to show that $V_0^0 = (S_0 - K)^+$. Since V_0^T is not defined at $T = 0$, we pass to the limit. We have three separate cases:

1. $S_0 = K$: Now $V_0^T = S_0 \Phi\left(\frac{\sigma}{2}\sqrt{T}\right) - S_0 \Phi\left(-\frac{\sigma}{2}\sqrt{T}\right)$ which tends to zero.

2. $S_0 > K$: Now $\log \frac{S_0}{K} > 0$ and thus $d_1(0, S_0)$ and $d_2(0, S_0)$ both tends to ∞. Hence, $V_0^0 = S_0 - K$.

3. $S_0 < K$: Now $\log \frac{S_0}{K} < 0$ and thus $d_1(0, S_0)$ and $d_2(0, S_0)$ both tends to $-\infty$. Hence, $V_0^0 = 0$.

We conclude that

$$\lim_{T\to 0+} V_0^T = (S_0 - K)^+$$

as it should.

Corollary 4.1.2. *Let S, $L_T^K(S)$, Φ, d_1, d_2 and V_0^T as in Theorem 4.1.1. Then the local time admits a representation*

$$\begin{aligned} L_T^K(S) &= 2\mu \int_0^T \left[\Phi(d_1(t,S_t)) - I_{S_t>K}\right] S_t \mathrm{d}t \\ &+ 2\sigma \int_0^T \left[\Phi(d_1(t,S_t)) - I_{S_t>K}\right] S_t \mathrm{d}W_t \\ &+ 2V_0^T - 2(S_0-K)^+. \end{aligned} \tag{4.3}$$

Proof. This follows simply by Definition (2.12) of the stochastic integral with respect to geometric Brownian motion. □

Remark. Note that

$$|\Phi(d_1(t,S_t)) - I_{S_t>K}| \le 1.$$

Thus the stochastic integral

$$\int_0^T \left[\Phi(d_1(t,S_t)) - I_{S_t>K}\right] S_t \mathrm{d}W_t \tag{4.4}$$

is a martingale.

4.2 On the expectation of the local time of geometric Brownian motion

In this section we discuss how the expectation of local time can be computed directly through the formula (2.28) and with use of representation (4.3).

Lemma 4.2.1. *Let X be a continuous semimartingale and $L_T^K(X)$ its local time at K. Then*

$$\mathbb{E}[L_T^K(X)] = \lim_{\epsilon\to 0+} \mathbb{E}\left[\frac{1}{\epsilon}\int_0^T I_{K\le X_u<K+\epsilon} \mathrm{d} < X, X >_u\right]. \tag{4.5}$$

Proof. In the proof of Lemma 2.3.9 we obtained

$$\frac{1}{\epsilon}\int_K^{K+\epsilon} L_T^a(X)\mathrm{d}a = \frac{1}{\epsilon}\int_0^T I_{K\le X_u<K+\epsilon}\mathrm{d} < X, X >_u$$

and the left hand side tends to $L_T^K(X)$. Take expectation on both sides and then let $\epsilon \to 0$. We have

$$\lim_{\epsilon\to 0+} \mathbb{E}\left[\frac{1}{\epsilon}\int_K^{K+\epsilon} L_T^a(X)\mathrm{d}a\right] = \lim_{\epsilon\to 0+} \mathbb{E}\left[\frac{1}{\epsilon}\int_0^T I_{K\le S_u<K+\epsilon}\mathrm{d} < X, X >_u\right].$$

In order to complete the proof, we need to show that

$$\lim_{\epsilon\to 0+} \mathbb{E}\left[\frac{1}{\epsilon}\int_K^{K+\epsilon} L_T^a(X)\mathrm{d}a\right] = \mathbb{E}\left[\lim_{\epsilon\to 0+}\frac{1}{\epsilon}\int_K^{K+\epsilon} L_T^a(X)\mathrm{d}a\right]. \tag{4.6}$$

This means that we can change the order of limit and integration and the result follows. But this is clear, since

$$\lim_{\epsilon\to 0+}\frac{1}{\epsilon}\int_K^{K+\epsilon} L_T^a(X)\mathrm{d}a = L_T^K(X).$$

On the other hand, local time is integrable for every a and if we take expectation and use Fubini's theorem, we obtain

$$\mathbb{E}\left[\frac{1}{\epsilon}\int_K^{K+\epsilon} L_T^a(X)\mathrm{d}a\right] = \frac{1}{\epsilon}\int_K^{K+\epsilon} \mathbb{E}[L_T^a(X)]\mathrm{d}a.$$

Now $\mathbb{E}[L_T^a(X)]$ is also a continuous function and by Lebesgue differentiation theorem we have

$$\lim_{\epsilon\to 0+}\frac{1}{\epsilon}\int_K^{K+\epsilon} \mathbb{E}[L_T^a(X)]\mathrm{d}a = \mathbb{E}[L_T^K(X)].$$

Hence (4.6) holds and we have the result. □

Let us apply this result for geometric Brownian motion.

Lemma 4.2.2. *Let S be geometric Brownian motion and $L_T^K(S)$ its local time at K. Put*

$$h_t(x) = \frac{1}{\sigma}\left[\log\frac{x}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)t\right].$$

Then

$$\mathbb{E}[L_T^K(S)] = \frac{\sigma S_0^2}{K}\int_0^T \phi_{2\sigma t,t}(h_t(K))e^{2\mu t+\sigma^2 t}\mathrm{d}t \tag{4.7}$$

for every $K > 0$, where $\phi_{a,b}$ is the probability density function of a normal distribution $N(a,b)$ i.e.

$$\phi_{a,b}(x) = \frac{1}{\sqrt{2\pi b}}e^{-\frac{(x-a)^2}{2b}}.$$

Proof. Recall that the quadratic variation of geometric Brownian motion satisfies

$$\mathrm{d} < S, S >_t = \sigma^2 S_t^2 \mathrm{d}t.$$

Subtitute this into the formula (4.5) to obtain

$$\mathbb{E}[L_T^K(S)] = \lim_{\epsilon\to 0+}\mathbb{E}\left[\frac{1}{\epsilon}\int_0^T I_{K\le S_t<K+\epsilon}\sigma^2 S_t^2\mathrm{d}t\right].$$

By Fubini's theorem we may change the order of integration and expectation. Thus we have

$$\mathbb{E}[L_T^K(S)] = \sigma^2\lim_{\epsilon\to 0+}\frac{1}{\epsilon}\int_0^T \mathbb{E}\left[I_{K\le S_t<K+\epsilon}S_t^2\right]\mathrm{d}t. \tag{4.8}$$

Next we need to calculate the expectation

$$\mathbb{E}\left[I_{K\le S_t<K+\epsilon}S_t^2\right].$$

First we observe that

$$K \le S_t < K+\epsilon \Leftrightarrow h_t(K) \le W_t < h_t(K+\epsilon).$$

Note that

$$\begin{aligned}\phi_{0,t}(x)e^{2\mu t+2\sigma x-\sigma^2 t} &= \tfrac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}e^{2\mu t+2\sigma x-\sigma^2 t}\\ &= \tfrac{1}{\sqrt{2\pi t}}e^{-\frac{(x-2\sigma t)^2}{2t}}e^{\sigma^2 t+2\mu t}\\ &= \phi_{2\sigma t,t}(x)e^{\sigma^2 t+2\mu t}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}\mathbb{E}\left[I_{K\le S_t<K+\epsilon}S_t^2\right] &= \textstyle\int_{\mathbb{R}}\phi_{0,t}(z)I_{h_t(K)\le z<h_t(K+\epsilon)}S_0^2 e^{(2\mu-\sigma^2)t}e^{2\sigma z}\mathrm{d}z\\ &= S_0^2 e^{(2\mu+\sigma^2)t}\textstyle\int_{h_t(K)}^{h_t(K+\epsilon)}\phi_{2\sigma t,t}(z)\mathrm{d}z.\end{aligned}$$

Substitute this into (4.8) to obtain

$$\mathbb{E}[L_T^K(S)] = \sigma^2 S_0^2 \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \int_0^T e^{(2\mu+\sigma^2)t} \int_{h_t(K)}^{h_t(K+\epsilon)} \phi_{2\sigma t,t}(z) \mathrm{d}z \mathrm{d}t.$$

Next we argue that we can change the order of limit and integration with respect to t. Note that if Y follows the normal distribution $N(2\sigma t, t)$, then

$$\int_{h_t(K)}^{h_t(K+\epsilon)} \phi_{2\sigma t,t}(z) \mathrm{d}z = \mathbb{P}(h_t(K) \leq Y \leq h_t(K+\epsilon)).$$

It is not difficult to show that, with fixed t,

$$\frac{1}{\epsilon} \mathbb{P}(h_t(K) \leq Y \leq h_t(K+\epsilon))$$

is bounded. This can be shown by use of Taylor series for example. Now we have that

$$e^{(2\mu+\sigma^2)t} \int_{h_t(K)}^{h_t(K+\epsilon)} \phi_{2\sigma t,t}(z) \mathrm{d}z$$

is dominated by an integrable function and thus, by Lebesgue dominated convergence theorem,

$$\mathbb{E}[L_T^K(S)] = \sigma^2 S_0^2 \int_0^T e^{(2\mu+\sigma^2)t} \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \int_{h_t(K)}^{h_t(K+\epsilon)} \phi_{2\sigma t,t}(z) \mathrm{d}z \mathrm{d}t.$$

Since $h_t(x)$ is differentiable with respect to x, we have

$$\lim_{\epsilon \to 0+} \frac{1}{\epsilon} \int_{h_t(K)}^{h_t(K+\epsilon)} \phi_{2\sigma t,t}(z) \mathrm{d}z = \phi_{2\sigma t,t}(h_t(K)) \frac{\mathrm{d}}{\mathrm{d}x} h_t(K)$$

by chain rule. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}x} h_t(K) = \frac{1}{\sigma K}.$$

Thus we have (4.7). □

As can be seen from equation (4.5), computing the expectation of local time can be quite tricky and requires numerical methods. Even for geometric Brownian motion, the integral in (4.7) requires numerical methods. Another way to compute the expectation of local time of geometric Brownian motion is to use representation

(4.3).

Lemma 4.2.3. *Let $h_t(x)$ be as in Lemma 4.2.2. Then the expectation of the local time $L_T^K(S)$ is given by*

$$\mathbb{E}[L_T^K(S)] = 2\int_0^T \mathbb{E}\left[\mu\left(\Phi(d_1(t,S_t)) - I_{S_t>K}\right)S_t\right]\mathrm{d}t + 2V_0^T - 2(S_0-K)^+. \quad (4.9)$$

Proof. Take expectation on both sides in (4.3) to obtain

$$\begin{aligned}\mathbb{E}[L_T^K(S)] &= 2\mathbb{E}\left[\int_0^T [\Phi(d_1^0(t,S_t)) - I_{S_t>K}]\,\mu S_t \mathrm{d}t\right] \\ &+ 2\mathbb{E}\left[\int_0^T [\Phi(d_1(t,S_t)) - I_{S_t>K}]\,\sigma S_t \mathrm{d}W_t\right] \\ &+ 2V_0^T - 2(S_0-K)^+.\end{aligned}$$

The expectation of the stochastic integral is zero. Moreover, by Fubini's theorem, we may change the order of expectation and integration on the first term and the result follows. □

The expectation $\mathbb{E}[\mu I_{S_t>K}S_t]$ can be computed and we have

$$\begin{aligned}\mathbb{E}[\mu I_{S_t>K}S_t] &= \mu S_0 \int_{h_t(K)}^\infty \phi_{0,t}(z) e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma z}\mathrm{d}z \\ &= \mu S_0 e^{\mu t}\int_{h_t(K)}^\infty \phi_{\sigma t,t}(z)\mathrm{d}z \\ &= \mu S_0 e^{\mu t}\left(1 - \Phi_{\sigma t,t}(h_t(K))\right).\end{aligned}$$

The expectation $\mathbb{E}[\mu\Phi(d_1(t,S_t))S_t]$ seems much more complicated. However, it turns out that the integral

$$\int_0^T \mathbb{E}[\mu\Phi(d_1(t,S_t))S_t]\mathrm{d}t$$

can be represented with $V_0^{T,\mu}$ and $V_0^{T,0}$.

Lemma 4.2.4. *Let S be geometric Brownian motion with drift $\mu = r$. Then*

$$\int_0^T \mathbb{E}[r\Phi(d_1(t,S_t))S_t]\mathrm{d}t = e^{rT}V_0^{T,r} - V_0^T. \quad (4.10)$$

Remark. Note that this gives also a relation between the price with interest rate r

and the price in the discounted model.

Proof. Now we work in a model with interest rate r. Recall that

$$\overline{V}_t = \overline{V}_0 + \int_0^t \gamma_u \mathrm{d}\overline{S}_u.$$

Now if we choose the portfolio to be the replicating portfolio for the European call, we get

$$e^{-rT}(S_T - K)^+ = V_0^{T,r} + \int_0^T v_x \mathrm{d}\overline{S}_t,$$

where v_x is the hedge. On the other hand, by Tanaka's formula we have

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T I_{S_t > K} \mathrm{d}S_t + \frac{1}{2} L_T^K(S).$$

Thus we obtain

$$e^{rT} V_0^{T,r} + e^{rT} \int_0^T v_x \mathrm{d}\overline{S}_t = (S_0 - K)^+ + \int_0^T I_{S_t > K} \mathrm{d}S_t + \frac{1}{2} L_T^K(S).$$

With respect to the martingale measure Q, the process $\overline{S}$ is a martingale and S is a geometric Brownian motion with drift $\mu = r$. Thus, by taking expectation with respect to Q, we obtain

$$e^{rT} V_0^{T,r} = (S_0 - K)^+ + r \int_0^T \mathbb{E}[I_{S_t > K} S_t]\mathrm{d}t + \frac{1}{2} \mathbb{E}[L_T^K(S)].$$

By substituting (4.9) we obtain

$$e^{rT} V_0^{T,r} = V_0^T + \int_0^T \mathbb{E}[r\Phi(d_1(t, S_t))S_t]\mathrm{d}t$$

and we have the result. □

With the use of this lemma we can improve our formula for the expectation of local time.

Theorem 4.2.5. *Let S be geometric Brownian motion with drift $\mu = r$ and $L_T^K(S)$*

its local time at K. *Let* $h_t(x)$ *be as in Lemma 4.2.2. Then*

$$\mathbb{E}[L_T^K(S)] = 2e^{rT}V_0^{T,r} - 2(S_0 - K)^+ - 2rS_0 \int_0^T e^{rt}\left(1 - \Phi_{\sigma t,t}(h_t(K))\right) \mathrm{d}t. \quad (4.11)$$

Proof. This follows directly by substituting (4.10) into (4.9). □

Remark. We have established two ways to compute the expectation of local time. However, both require numerical methods. Note also that the drift term occurs in different places on (4.11) and (4.7), yet they are the same.

4.3 Applications

In this section we introduce some applications of the representations (4.1) and (4.3). First we study a few special cases. We analyse the local time of exponential martingale and the local time of geometric Brownian motion at starting point S_0. We also consider the consequences on Black and Scholes differential equation (3.13).

4.3.1 The local time of exponential martingale

Geometric Brownian motion with drift $\mu = r$ is an exponential martingale in the discounted model $r = 0$. Now S is an exponential martingale and the the expectation of local time (4.11) takes the form

$$\mathbb{E}[L_T^K(S)] = 2V_0^T - 2(S_0 - K)^+. \quad (4.12)$$

This is a simple way to compute the expectation of local time of exponential martingale. Moreover, we conclude that when $\mu = 0$, the integral in (4.7) is given by

$$2V_0^T - 2(S_0 - K)^+.$$

Recall also that the local time is an increasing process. Thus, for every $t > s$, it holds

$$\mathbb{E}[L_t^K(S)] \geq \mathbb{E}[L_s^K(S)].$$

According to (4.12), this is equivalent to

$$V_0^t \geq V_0^s$$

which means that in the discounted model the initial capital V_0^T of the replicating portfolio of the European call is an increasing function in maturity T. This can also be computed directly by differentiating (4.2). In economical terms this means that when the maturity of an option is higher, the price of the option is higher. Intuitively this can be explained as when the maturity date of the option increases, the risk included increases also. In order to hedge this risk one needs to invest more capital. However, this is not true for every European option as is shown in section 4.4.

4.3.2 The local time at the starting point

Recall that the local time $L_T^K(S)$ is related to the amount of time S spends on the level K up to T. Let us discuss the local time $L_T^{S_0}(S)$ of geometric Brownian motion at the starting point S_0. Now $K = S_0$, and V_0^T is of the form

$$V_0^T = S_0 \Phi\left(\frac{1}{2}\sigma\sqrt{T}\right) - S_0 \Phi\left(-\frac{1}{2}\sigma\sqrt{T}\right) = 2S_0 \Phi\left(\frac{1}{2}\sigma\sqrt{T}\right) - S_0$$

by the symmetry of standard normal distribution. Thus, the representation (4.1) takes the form

$$L_T^{S_0}(S) = 2\int_0^T [\Phi(d_1(t, S_t)) - I_{S_t > S_0}]\, \mathrm{d}S_t + 4S_0 \Phi\left(\frac{1}{2}\sigma\sqrt{T}\right) - 2S_0.$$

If we take expectation with respect to the measure Q we obtain

$$\mathbb{E}_Q[L_T^{S_0}(S)] = 4S_0 \Phi\left(\frac{1}{2}\sigma\sqrt{T}\right) - 2S_0$$

for every $T < \infty$. Recall that with respect to Q the process S is the exponential martingale. Despite how big the maturity T is, we still have

$$\mathbb{E}[L_T^{S_0}(S)] < 2S_0$$

for exponential martingale. This means that despite the time T, the expectation of local time of exponential martingale at the starting level S_0 is bounded by $2S_0$. This is intuitive, since the exponential martingale tends to zero as time tends to infinity.

4.3.3 On the Black and Scholes differential equation

Consider the differential equation (3.13) with $f(x) = (x - K)^+$ and $r = 0$. We know that if v is the solution for the differential equation, then v_x is the hedge for European call options. On the other hand, we know that $V_0^T = v(0, S_0)$ is an increasing function in T. Thus, when the time interval on the differential equation is longer, then the boundary values $v(0, S_0)$ are larger. This is an interesting result, since the differential equation assumes nothing about the values at $t = 0$ and yet we can say something about them. We may also compute the values $v(0, S_0)$ for any positive S_0 from equation (4.12), if we know the expected value of the local time. In addition, we can compute the expected value of local time numerically.

4.4 Convex European options

In this section we consider European options $f(S_T)$ where f is a difference of two convex function and derive a formula for the price of such options in the discounted model $r = 0$. We denote the price of the option f by V_f^T and the price of the call option with strike price K by $V_0^T(K)$.

Theorem 4.4.1. *Let f be a difference of two convex functions and $f(S_T)$ the corresponding European option with maturity T. If*

$$\mathbb{E}\left[\int_{\mathbb{R}_+} L_T^a(S) f^{\cdot\cdot}(\mathrm{d}a)\right] < \infty, \tag{4.13}$$

then the price V_f^T of the option is given by

$$V_f^T = f(S_0) + \int_{\mathbb{R}_+} V_0^T(a) - (S_0 - a)^+ f^{\cdot\cdot}(\mathrm{d}a). \tag{4.14}$$

Proof. Recall that now, again, we work in a discounted model $r = 0$. Thus we have

$$f(S_T) = V_f^T + \int_0^T v_x \mathrm{d}S_u,$$

where v_x is the hedge of the claim. On the other hand, by Theorem 2.3.6 we have

$$f(S_T) = f(S_0) + \int_0^T f^*(S_u)\mathrm{d}S_u + \frac{1}{2}\int_{\mathbb{R}} L_T^a(S) f^{\cdot\cdot}(\mathrm{d}a).$$

The last integral can be taken over the positive real line since the local time of a positive semimartingale at the level a equals zero for every negative a. From these equations we conclude

$$V_f^T = f(S_0) + \int_0^T f^*(S_u) - v_x \mathrm{d}S_u + \frac{1}{2}\int_{\mathbb{R}_+} L_T^a(S) f^{\cdot\cdot}(\mathrm{d}a). \tag{4.15}$$

Take expectation with respect to the measure Q. Then the expectation of stochastic integral is zero and we have

$$V_f^T = f(S_0) + \frac{1}{2}\mathbb{E}_Q\left[\int_{\mathbb{R}_+} L_T^a(S) f^{\cdot\cdot}(\mathrm{d}a)\right].$$

By Fubini's theorem and equation (4.12) we have

$$V_f^T = f(S_0) + \int_{\mathbb{R}_+} V_0^T(a) - (S_0 - a)^+ f^{\cdot\cdot}(\mathrm{d}a).$$

Hence we have (4.14). □

Remark. In the proof we used the Fubini's theorem and for that we need the condition (4.13). However, this is not a very limiting condition. In practice, the integral is finite for every option which are meaningless in a sense that the price is finite. For this we needed the square integrability discussed in chapter 3.

Now we have obtained a formula for the price of a European option which is determined by a difference of two convex function. As a corollary, we have that the price of a convex option increases as the maturity increases and the price of a concave option decreases as the maturity increases.

Corollary 4.4.2. *Let $f(S_T)$ be a European option which satisfies the assumptions* *of Theorem 4.4.1. Then the price of the option is increasing in T for convex f and decreasing in T for concave f.*

Proof. By equation (4.14) we have

$$V_f^t - V_f^s = \int_{\mathbb{R}_+} V_0^t(a) - V_0^s(a) f^{\cdot\cdot}(\mathrm{d}a).$$

Recall that the price of a European call is increasing in maturity i.e.

$$V_0^t(a) - V_0^s(a) \geq 0.$$

The result follows from the fact that $f^{\cdot\cdot}$ is a positive measure for convex f and negative measure for concave f. □

Remark. One could expect that the price of an option increase in maturity since the fluctuation of the price can be wider in larger time interval. As the result shows, this is indeed so in a convex case. However, the fact that the price decreases in maturity for concave options means that this intuitive explanation is not correct in all cases.

Example. Consider an European put option with strike price K and maturity T. Now $f(x) = (K - x)^+$ and $f^{\cdot\cdot}$ gives the mass one to the point K and zero for the rest. Thus

$$\int_{\mathbb{R}_+} V_0^T(a) - (S_0 - a)^+ f^{\cdot\cdot}(\mathrm{d}a) = V_0^T(K) - (S_0 - K)^+$$

and we have

$$V_0^{put} = (K - S_0)^+ + V_0^{call} - (S_0 - K)^+.$$

Observe that

$$(K - S_0)^+ - (S_0 - K)^+ = (K - S_0)^+ - (K - S_0)^- = K - S_0$$

which yields

$$V_0^{put} = V_0^{call} + K - S_0.$$

Thus we obtained the put-call-parity in the discounted model.

Chapter 5

Conclusions

In this study we have established the connection between the local time of geometric Brownian motion and European call option in the Black and Scholes model. We have also applicated results for more general European options. First we have reviewed the details of the Black and Scholes model. We have introduced the assumptions of the model and shown the basic properties such as completeness and arbitrage-free property of the model. We have also introduced the basic stochastic analysis which the model is based on.

As the result, we have derived a new integral representation for the local time of geometric Brownian motion through the Black and Scholes model and we have derived a price formula for the price of European options which are determined by the difference of two convex functions. According to the integral representation, the local time of geometric Brownian motion can be represented as a linear combination of a stochastic integral with respect to geometric Brownian motion, the return of the European call at start and the Black and Scholes price of European call option. We have also found two ways to compute the expectation of the local time. We have computed the expectation directly and through the new representation.

We have introduced applications and consequences of the results. We have applicated the integral representation of local time for the local time of exponential martingale and in this case, we have found a simple formula for the expectation of local time of exponential martingale. We have also shown that the expectation of the local time of exponential martingale on the starting level is bounded for any time.

As a financial application, we have derived the price formula which gives a relation between the price of European options which are determined by the difference of two convex functions and the price of the European call. We have also shown that the price of a convex European option increases as the maturity of the option increases and the price of a concave European option decreases as the maturity of the option increases.

We have analysed the influence of the results to the deterministic Black and Scholes differential equation. We have shown that as the time interval becomes wider, the boundary values of the solution on the beginning of the time interval increases. This is interesting, since the differential equation assumes nothing about the boundary values on the beginning and yet we may say something about these values. Our integral representation offers also a way to compute the boundary values of the solution, if we know the expectation of local time.

The results open a few interesting directions of further study. First of all, in finance it is of interest how sensitive the price of an option is with respect to the parameters such as maturity, volatility and interest rate. These so-called greeks are well known for call option and thus it would be interesting to study, whether the price formula offers a way for similar sensitivity analysis for more general options. Secondly, the differential equation would be also of great interest, if the boundary condition is allowed to be any convex or concave function.

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Appendix A

The expectation of e^{cZ} where $Z \sim N(\mu, \sigma^2)$

Let us compute the expectation of e^{cZ}, where c is a constant and $Z \sim N(\mu, \sigma^2)$. Let $\phi_{a,b}$ be the probability density function of the normal distribution $N(a, b)$.

$$
\begin{aligned}
\mathbb{E}[e^{cZ}] &= \int_{\mathbb{R}} \phi_{\mu,\sigma^2}(z) e^{cz} \mathrm{d}z \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} e^{cz} \mathrm{d}z \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(z^2 - 2\mu z + \mu^2 - 2cz\sigma^2)} \mathrm{d}z \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(z^2 - 2(\mu + c\sigma^2)z + (\mu + c\sigma^2)^2)} e^{-\frac{1}{2\sigma^2}(\mu^2 - (\mu + c\sigma^2)^2)} \mathrm{d}z \\
&= e^{-\frac{1}{2\sigma^2}(\mu^2 - (\mu + c\sigma^2)^2)} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{[z - (-\mu + c\sigma^2)]^2}{2\sigma^2}} \mathrm{d}z \\
&= e^{-\frac{1}{2\sigma^2}(\mu^2 - (\mu + c\sigma^2)^2)} \int_{\mathbb{R}} \phi_{\mu + c\sigma^2, \sigma^2}(z) \mathrm{d}z \\
&= e^{\frac{1}{2}\sigma^2 c^2 + \mu c}.
\end{aligned}
$$

Appendix B

On the value function of European options

In section 3.4, we computed the value function for square integrable European options and used the fact that it is in $C^{1,2}([0,T),\mathbb{R})$. In computations, we used the result (2.4) and changed the order of derivation and integration without hesitation. Here we go through technical details, why these operations are justified. We had that

$$v(t,x) = \int_{\mathbb{R}_+} f(z)k(z,x,\theta)\mathrm{d}z,$$

where the kernel k is given by

$$k(z,x,\theta) = \frac{e^{-r\theta}}{\sigma z\sqrt{2\pi\theta}} e^{-\frac{1}{2\sigma^2\theta}\left[\log\frac{z}{x}-\left(r-\frac{\sigma^2}{2}\right)\theta\right]^2}.$$

First of all, note that

$$\begin{aligned}
\mathbb{E}[f(S_T)^2] &= \mathbb{E}[f^2(S_0 e^{rT-\frac{\sigma^2}{2}T+\sigma W_T})] \\
&= \mathbb{E}[f^2(S_t e^{r(T-t)-\frac{\sigma^2}{2}(T-t)+\sigma(W_T-W_t)})] \\
&= \textstyle\int_{\mathbb{R}^2} f^2(y e^{r(T-t)-\frac{\sigma^2}{2}(T-t)+\sigma x})\mathbb{P}(S_t\in \mathrm{d}y)\mathbb{P}(W_T-W_t\in\mathrm{d}x) \\
&= \textstyle\int_{\mathbb{R}} f^2(S_0 e^{rT-\frac{\sigma^2}{2}T+\sigma x})\mathbb{P}(W_T\in\mathrm{d}x) \\
&= \textstyle\int_{\mathbb{R}} f^2(S_0 e^{rT-\frac{\sigma^2}{2}T+\sigma x})\frac{1}{\sqrt{2\pi T}}e^{-\frac{x^2}{2T}}\mathrm{d}x \\
&= \textstyle\int_{\mathbb{R}} f^2(S_0 e^{rT-\frac{\sigma^2}{2}T+\sigma\sqrt{T}x})\mathbb{P}(Z\in\mathrm{d}x) \\
&= \textstyle\int_{\mathbb{R}} f^2(S_0 e^{rT-\frac{\sigma^2}{2}T+\sigma\sqrt{T}x})\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\mathrm{d}x.
\end{aligned}$$

Since our claim is square integrable, these are all finite. For the result we need the Fubini's theorem. Put

$$g(S_t, W_T - W_t) = f(S_t e^{r(T-t)-\frac{\sigma^2}{2}(T-t)+\sigma(W_T-W_t)}).$$

Now we have to check that

$$\int_{\mathbb{R}^2} |g(S_t, W_T - W_t)| \mathbb{P}(S_t \in \mathrm{d}y) \mathbb{P}(W_T - W_t \in \mathrm{d}x) < \infty.$$

Let N denote the set $\{|g| \le 1\}$. We have that

$$\begin{aligned} & \textstyle\int_{\mathbb{R}^2} |g(x,y)| \mathbb{P}(S_t \in \mathrm{d}x) \mathbb{P}(W_T - W_t \in \mathrm{d}y) \\ = & \textstyle\int_N |g(x,y)| \mathbb{P}(S_t \in \mathrm{d}x) \mathbb{P}(W_T - W_t \in \mathrm{d}y) \\ + & \textstyle\int_{\mathbb{R}^2 - N} |g(x,y)| \mathbb{P}(S_t \in \mathrm{d}x) \mathbb{P}(W_T - W_t \in \mathrm{d}y) \\ \le & \textstyle 1 + \int_{\mathbb{R}^2 - N} g^2(x,y) \mathbb{P}(S_t \in \mathrm{d}x) \mathbb{P}(W_T - W_t \in \mathrm{d}y) < \infty. \end{aligned}$$

Here we used the fact that f is square integrable. Thus the Fubini's theorem is justified. Note also that since

$$\int_{\mathbb{R}^2} |g(x,y)| \mathbb{P}(W_T - W_t \in \mathrm{d}y) \mathbb{P}(S_t \in \mathrm{d}x) < \infty,$$

we have that

$$\int_{\mathbb{R}} |g(x,y)| \mathbb{P}(W_T - W_t \in \mathrm{d}y) < \infty$$

for almost every x. On the other hand, v can be represented in the form

$$v(t,x) = \int_{\mathbb{R}} g(x,y) \mathbb{P}(W_T - W_t \in \mathrm{d}y).$$

Thus v is finite for almost every x. On the other hand, from the form

$$v(t,x) = \int_{\mathbb{R}_+} f(z) k(z,x,\theta) \mathrm{d}z$$

we see that v is continuous. Thus it is finite for every x. Next we argue that the value function is smooth. Kernel k is continuously differentiable with its parameteres. Thus the function v is also continuously differentiable, if we can change the order of integration and derivation. Let us first show that the partial derivative v_x exists. Fix x and θ. The difference quotient is given by

$$\int_{\mathbb{R}_+} f(z) \cdot \frac{k(z, x+\epsilon, \theta) - k(z, x, \theta)}{\epsilon} \mathrm{d}z.$$

We want to apply dominated convergence theorem. If we take ϵ small enough, an upper bound for the difference quotient can be given by

$$\frac{k(z, x+\epsilon, \theta) - k(z, x, \theta)}{\epsilon} \leq \sup_{c \in [x, x+1]} |k_x(z, c, \theta)|.$$

Now we only need to show that

$$\int_{\mathbb{R}_+} f(z) \sup_{c \in [x, x+1]} |k_x(z, c, \theta)| \mathrm{d}z < \infty.$$

The integral can be given as

$$\begin{aligned} \textstyle\int_{\mathbb{R}_+} f(z) \sup_{c \in [x,x+1]} |k_x(z,c,\theta)| \mathrm{d}z &= \textstyle\int_0^a f(z) \sup_{c \in [x,x+1]} |k_x(z,c,\theta)| \mathrm{d}z \\ + \textstyle\int_a^N f(z) \sup_{c \in [x,x+1]} |k_x(z,c,\theta)| \mathrm{d}z &+ \textstyle\int_N^\infty f(z) \sup_{c \in [x,x+1]} |k_x(z,c,\theta)| \mathrm{d}z. \end{aligned}$$

The middle term does not cause any problems. Thus we only need to check that we can fix a and N such that the first and third integrals are finite. By direct computation we obtain

$$k_x(z, x, \theta) = \frac{1}{\sigma^2 \theta} k(z, x, \theta) \cdot \frac{\log \frac{z}{x} - \left(r - \frac{\sigma^2}{2}\right) \theta}{x}.$$

Moreover,

$$\sup_{c \in [x, x+1]} |k_x(z, c, \theta) \leq \frac{1}{x \sigma^2 \theta} \sup_{c \in [x, x+1]} k(z, c, \theta) \cdot | \left(\log \frac{z}{c} - \left(r - \frac{\sigma^2}{2}\right) \theta \right) |.$$

Let us next find the supremum

$$\sup_{c \in [x, x+1]} k(z, c, \theta) \cdot | \left(\log \frac{z}{c} - \left(r - \frac{\sigma^2}{2}\right) \theta \right) |.$$

By continuity, the supremum equals maximum since the interval is closed. Thus we only need to find the maximum. This can be done by forgetting absolute value and finding all extreme values. In addition, we can take a and N such that $g < 0$ for every $z < a$ and $g > 0$ for every $z > N$. Extreme values can be found on end points of the interval or on the points where derivative equals zero. Put

$$g(z, x, \theta) = \log \frac{z}{x} - \left(r - \frac{\sigma^2}{2} \right) \theta.$$

Then

$$\begin{aligned} & \tfrac{\mathrm{d}}{\mathrm{d}c} k(z, c, \theta) \cdot \left(\log \tfrac{z}{c} - \left(r - \tfrac{\sigma^2}{2} \right) \theta \right) \\ = \; & \tfrac{1}{\sigma^2 \theta} k(z, c, \theta) \tfrac{g^2(z,c,\theta)}{c} - \tfrac{k(z,c,\theta)}{c}. \end{aligned}$$

This equals zero exactly when

$$x = z e^{\pm\sqrt{\sigma^2\theta} - \left(r - \frac{\sigma^2}{2}\right)\theta}.$$

Next we take a such small that neither of these zeros cannot be achieved for any $z \in (0, a)$ and thus the extreme values are achieved on point x or $x + 1$. In similar way, we take N so large that neither of these zeros cannot be achieved for any $z \in (N, \infty)$. Thus the potential extreme values are on x or $x + 1$. Now we have deduced that

$$\begin{aligned} & \textstyle\int f(z) \sup_{c \in [x, x+1]} |k_x(z, c, \theta)| \mathrm{d}z \\ \leq \; & h(x, \theta) \textstyle\int f(z) \sup_{c \in [x, x+1]} k(z, c, \theta) |g(z, c, \theta)| \mathrm{d}z. \end{aligned}$$

Here the integral is the integral over an interval $(0, a)$ or (N, ∞) and h is only a short notation for a function which is a constant, since x and θ was fixed. As explained above, we can forget the absolute value of g and check that these integral are finite for $c = x$ and $c = x + 1$. Depending on this, we only multiply and divide by x or $x + 1$ and we have the form

$$\overline{x} h(x, \theta) \int f(z) k_x(z, \overline{x}, \theta) \mathrm{d}z$$

where $\overline{x} = x$ or $\overline{x} = x + 1$. This implies that we only need to show that

$$\int_{\mathbb{R}_+} f(z)k_x(z,x,\theta)\mathrm{d}z < \infty.$$

By the use of Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}_+} f(z)k_x(z,x,\theta)\mathrm{d}z \\
= \;& \tfrac{1}{\sigma^2\theta}\textstyle\int_{\mathbb{R}_+} f(z)k(z,x,\theta)\frac{g(z,x,\theta}{x}\mathrm{d}z \\
\leq \;& \tfrac{1}{\sigma^2\theta}\sqrt{\textstyle\int_{\mathbb{R}_+} f^2(z)k(z,x,\theta)\mathrm{d}z}\sqrt{\textstyle\int_{\mathbb{R}_+} k(z,x,\theta)\frac{\left(\log\frac{z}{x}-\left(r-\frac{\sigma^2}{2}\right)\theta\right)^2}{x^2}\mathrm{d}z}.
\end{aligned}$$

The first integral is finite, since f is square integrable. Making a change of variable we obtain that the second integral is of the form

$$\int_{\mathbb{R}} Ce^{-y^2}y^2\mathrm{d}y$$

where C is a constant depending on x and θ. Since this integral is finite, we conclude that we can change the order of integration and derivation. Thus the partial derivative v_x of value function is given by

$$v_x(t,x) = \int_{\mathbb{R}_+} f(z)k_x(z,x,\theta)\mathrm{d}z.$$

With similar arguments, one can conclude that partial derivatives v_{xx} and v_t exists exactly when

$$\int_{\mathbb{R}_+} f(z)k_{xx}(z,x,\theta)\mathrm{d}z < \infty$$

and

$$\int_{\mathbb{R}_+} f(z)k_\theta(z,x,\theta)\mathrm{d}z < \infty.$$

These can be shown in a similar way. We omit the details.

Appendix C

The value process for European call option

Let us compute the value process given by (3.8) in the case of European call option $f(x) = (x-K)^+$. Define d_1 and d_2 as

$$\begin{aligned} d_1(t,x) &:= \tfrac{\log \frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2(t,x) &:= d_1(t,x) - \sigma\sqrt{T-t}. \end{aligned}$$

Moreover, let $\Phi(x)$ be the cumulative distribution function of the normal distribution $N(0,1)$.

$$\begin{aligned} v(t,x) &= e^{-r(T-t)} \int_{\mathbb{R}} f(xe^{r(T-t)+\sigma\sqrt{T-t}z-\frac{\sigma^2}{2}(T-t)}) \tfrac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z \\ &= \int_{\mathbb{R}} (xe^{\sigma\sqrt{T-t}z-\frac{\sigma^2}{2}(T-t)} - Ke^{-(T-t)})^+ \tfrac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z. \end{aligned}$$

Clearly

$$\begin{aligned} xe^{\sigma\sqrt{T-t}z-\frac{\sigma^2}{2}(T-t)} &\geq Ke^{-r(T-t)} \\ \Leftrightarrow \quad \sigma\sqrt{T-t}z - \tfrac{\sigma^2}{2}(T-t) &\geq \ln\tfrac{K}{x} - r(T-t) \\ \Leftrightarrow \quad z &\geq \tfrac{1}{\sigma\sqrt{T-t}}\left(\tfrac{\sigma^2}{2}(T-t) - r(T-t)\right) \\ \Leftrightarrow \quad z &\geq -d_2(t,x). \end{aligned}$$

Thus

$$\begin{aligned} & \int_{\mathbb{R}} (xe^{\sigma\sqrt{T-t}z-\frac{\sigma^2}{2}(T-t)} - Ke^{-(T-t)})^+ \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z \\ = & \int_{-d_2(t,x)}^{\infty} (xe^{\sigma\sqrt{T-t}z-\frac{\sigma^2}{2}(T-t)} - Ke^{-(T-t)}) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z. \end{aligned}$$

By symmetry of the normal distribution we have

$$\begin{aligned} & \int_{-d_2(t,x)}^{\infty} (xe^{\sigma\sqrt{T-t}z-\frac{\sigma^2}{2}(T-t)} - Ke^{-(T-t)}) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z \\ = & \int_{-\infty}^{d_2(t,x)} (xe^{-\sigma\sqrt{T-t}z-\frac{\sigma^2}{2}(T-t)} - Ke^{-(T-t)}) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z \\ = & \, x \int_{-\infty}^{d_2(t,x)} xe^{-\sigma\sqrt{T-t}z-\frac{\sigma^2}{2}(T-t)} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z \\ - & \int_{-\infty}^{d_2(t,x)} Ke^{-(T-t)} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z \\ = & \, x \int_{-\infty}^{d_2(t,x)} xe^{-\sigma\sqrt{T-t}z-\frac{\sigma^2}{2}(T-t)} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z \\ - & \, Ke^{-(T-t)}\Phi(d_2(t,x)). \end{aligned}$$

Put $y = z + \sigma\sqrt{T-t}$ in the first integral. Then $\mathrm{d}y = \mathrm{d}z$ and

$$z^2 = (y - \sigma\sqrt{T-t})^2 = y^2 - 2\sigma\sqrt{T-t}y + \sigma^2(T-t).$$

Moreover,

$$z \leq d_2(t,x) \Leftrightarrow y \leq d_1(t,x).$$

We get

$$\begin{aligned} & \int_{-\infty}^{d_2(t,x)} xe^{-\sigma\sqrt{T-t}z-\frac{\sigma^2}{2}(T-t)} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z \\ = & \int_{-\infty}^{d_1(t,x)} xe^{-\sigma\sqrt{T-t}y+\sigma^2(T-t)-\frac{\sigma^2}{2}(T-t)+\sigma\sqrt{T-t}y-\frac{\sigma^2}{2}(T-t)} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \mathrm{d}y \\ = & \, x\Phi(d_1(t,x)). \end{aligned}$$

Thus the value function (3.8) for European call is given by

$$v(t, S_t) = S_t\Phi(d_1(t, S_t)) - Ke^{-(T-t)}\Phi(d_2(t, S_t)).$$